Exercises II

22/3 2012

Due 2/4

1 Find a necessary and sufficient condition on the real number b such that

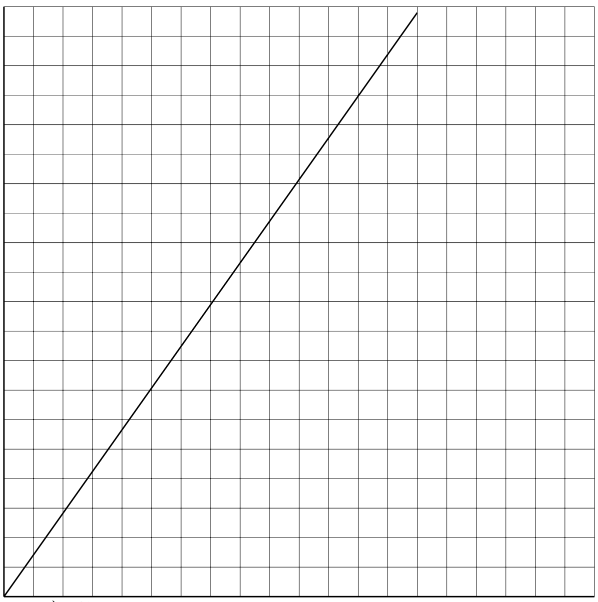
$$\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$$

2 Let α be a rational number. Show that there is only a finite number of possibilities for $\{n\alpha\}$, while if α is irrational there is an infinity of possibilities. Show in fact that $\{n\alpha\}$ can be arbitrarily small, and hence the set $\{n\alpha\}$ is dense in the interval $(0, 1)$

3 Show that the number ${}^{10}\log 2$ is irrational, and conclude that any sequence of numbers 7325..6 can start 2^m. In particular find the smallest m such that 2^m starts with either 7 or 9

4 In the sequence $1, 2, 4, 5, 7, 8, 9 \ldots \lfloor n\sqrt{2} \rfloor$ we have either sequences of consequetive numbers of length 2 $(1, 2$ or $4, 5)$ or of length 3 $(7, 8, 9)$. Show that the length of consequetive numbers is either two or three. If we code the above sequence as $22323223\ldots$ can you see any obvious patterns? Such as periodicity or any other constraints?

5 Consider the grid given by the lines $x = n$ and $y = m$ for arbitrary intergers n, m. Then consider a line with slope α. This line will intersect horizontal and vertical lines, and we can encode this by a sequence $HVHV\ldots$. In the case $\alpha = p/q$ is rational, the lines will go through intersection points (also known as lattice points of the lattice $\mathbb{Z}$) at regular intervals.



a) How many line-intersections will we have before the first hit of a lattice points.

b) Show that the sequence $HV \ldots VH$ will be palindromic in this case.

If α is not rational, then the line will never hit a lattice point, and we will have an infinite sequence of letters H, V.

c) Show that this sequence is never periodic, and that there are restrictions on the number of consequetive H and V respectively.

d) In case $\alpha = \sqrt{2}$ relate the sequence to the one in the previous exercise.

6 Show that the sequence $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$ is given by $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$ where the number m occurs m times.