Övningsproblem

1.1. 29/3 2012

1 Solve the congruence $x = 2(3), x = 3(5), x = 4(7)$

Consider $n = a \times 3 \times 5 + b \times 3 \times 7 + c \times 5 \times 7$. We need to solve thhe congruences $15a = 4(7), 21b = 3(5)$ and $35c = 2(3)$. Those are easily reduced to $a = 4(7), b = 3(5)$ and $-c = -1(3)$. Thus $a = 4, b = 3, c = 1$ solves it. hence we can choose $n = 60 + 63 + 35 = 158$. We can subtract multiples of $3 \times 5 \times 7 = 105$ hence 53 is the smallest positive solution, and -52 the one closest to zero.

2 Compute the number $\psi(210)$ **You need to factorize** 210 **first**

We find immediately that $210 = 2 \times 3 \times 5 \times 7$, hence

$$\psi(210) = (2-1)(3-1)(5-1)(7-1) = 48$$

3 Find all integers n **such that** $\psi(n) = 4$

We can write 4 as 4 or 2×2. In the first case we my look for either a primes p such that $p - 1 = 4$ i.e. $p = 5$, or such that $p^n - p^{n-1} = 4$. The latter forces $p = 2$ with the unique solution $2^3 = 8$. We may also in the case of $n = 5$ throw in the single factor of 2 getting $n = 10$. In the second case we may look at both $p - 1 = 2$ and $p^n - p^{n-1} = 2$ with the unque solution $3 \times 2^2 = 12$. Thus $5, 8, 10$ and 12.

4 Find the smallest integer n **with exactly twelve divisors, including** 1 **and the number** n **itself.**

If $n = \prod_i p_i^{n_i}$ the number of divisors is given by $\prod_i (n_i + 1)$. It is clear that we would like to factor 12 in as many factors as possible. The maximum number is given by $3 \times 2 \times 2$ which corresponds to $p_0^2 p_1 p_2$. The most economical way is to chose $2^2 \times 3 \times 5 = 60$ the Old Babylonian solution.

5 Show that if $n > 4$ **then** $n!$ **ends with a zero. How many zero does** 100! **end with? Would you be able to determine the first digit of** 100!**. Finally give the smallest prime** p **which does not divide** 100!**. Can you compute its residue modulo that prime?** If $n > 4$ then $n!$ contains the factors 2 and 5 and it becomes obvious. For the second we need to compute the maximal power of 5 in 100! which surely will be matched by a corresponding power of 2. There are $100/5 = 20$ factors divisible by 5 and $100/25 = 4$ factors divisible by 25 and none divisible by 125. Hence there are $20 + 4$ factors of 5 corresponding to 24 zeroes.

If we make the trapetzoidal approximation of $\int_1^{100} \log x$ we will get

$$\sum_n^{99} \log n + \frac{1}{2}\log 100 = \int_1^{100} \log x = x \log x - x|_{x=1}^{x=100} = 100 \log 100 - 99 + \epsilon$$

where ϵ is the error. We need to be able to compute the sum $\sum_{n=1}^{100} \log n$ with sufficient accuracy, in order to infer the first digit from its fractional part.

Now $\log x$ is concave and lies above its piecewise linear approximation. We consider for each piece $\psi_n(x) = \log x - (\log(n+1) - \log n)x + n(\log(n+1) - \log n) -$

$\log n$ and try to find the maximum. It is given by $x_n = 1/(\log(n+1) - \log n)$. We easily compute $\log(n+1) - \log n = \log n(1+\frac{1}{n}) - \log n = \log(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \dots$ and hence we get

$$x_n = n(1+(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}+\dots)+(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}+\dots)^2+(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}+\dots)^3+\dots)$$

which simplifies to

$$x_n = n(1 + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{24n^3} + \dots)$$

Hence

$$\begin{aligned} \log x_n = & \log n + (\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{24n^3} + \dots) \\ & - \frac{1}{2}(\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{24n^3} + \dots)^2 \\ & + \frac{1}{3}(\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{24n^3} + \dots)^3 + \dots \end{aligned}$$

which simplifies to

$$\log(x_n) = \log n + \frac{1}{2n} - \frac{5}{24n^2} + \frac{1}{8n^3} + \dots$$

We now plug in the value of x_n in $\psi_n(x)$ and get

$$\log n + \frac{1}{2n} - \frac{5}{24n^2} + \frac{1}{8n^3} + \dots - 1 + n(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots) - \log n$$

which simplifies to

$$\frac{1}{8n^2} - \frac{1}{4n^3} + \dots$$

Summing this over all n gives us a bound on the total error, which is about twice as large as the real. For small values of n the error is significant, so the best strategy would be to compute say the first ten terms separately and then use the remaining estimate which will be somewhat big. A better error estimate can be given, but it involves some further work.

Now if we look at $\sum_{n=N}^{\infty} \frac{1}{n^2}$ this is approximately $\int_N^{\infty} \frac{dt}{t^2} = -\frac{1}{n}|_{n=N}^{\infty} = \frac{1}{N}$. This means that the error will be bounded by $1/80$ if we start at $N = 10$. Instead of using natural logarithms we will use base ten. This means that we multiply everything with $c = \log_{10} e = 0.434292\dots$.

Thus we look at

$$\log 1 + \log 2 + \log 3 + \dots + \log 9 + (\log 10 + \dots \log 100)$$

where the final tail can be approximated by

$$\begin{aligned} \int_{10}^{100} \log(t)dt = c(t\ln(t) - t)|_{t=10}^{t=100} &= 100\log(100) - 10\log(10) - c(100-10) \\ &= 190 - c(90) = 190 - 39.086279 = 150.913721.. \end{aligned}$$

. In fact we have that the integral is

$$\frac{1}{2}\log 10 + \log 11 + \log 12 + \cdots + \log 98 + \log 99 + \frac{1}{2}\log 100$$

up to an error which we estimate to $c\frac{1}{8}\frac{1}{10} = 0.05$. Now we can compute

$$\begin{aligned}\sum_{n=1}^{100}\log(n) &= (\log 1 + \log 2 + \cdots + \log 9) \\ +(\frac{1}{2}\log 10 + \log 11 + \cdots + \log 99 + \frac{1}{2}\log 100) &+ (\frac{1}{2}\log 10 + \frac{1}{2}\log 100) \\ &= 5.559763 + 150.913721 + 1.5 \\ &= 157.974484\end{aligned}$$

The middle sum is overestimated by something that is on the order av 0.05 thus we should expect that 100! starts with 8 or 9 as $\log(8) = 0.903090$ and $\log(9) = 0.954243$. In order to determine which we need to make a better error estimate.

In fact a brute force computation using numbers encoded in large arrays produces

$$\begin{aligned}&93326215443944152681699238856266700490715968264381621468592963895217\\&5999932299156089414639761565182862536979208272237582511852109168640\\&00000000000000000000000\end{aligned}$$

which indeed ends with the predicted 24 zeroes and which also contains 158 digits.

6 Write down the Farey sequence F_N for $N = 7$

$$\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$$