Exercises IV

23/4 2012

Due 30/4

1 Show that $\binom{2n}{n}$ is always even. What can you say about $\binom{2n\pm 1}{n}$?

2 Work out Pascal's triangle modulo two. Can you see any pattern?

3 Compute using a computer $\left\{ {15 \atop 6} \right\}$ and $\left[{12 \atop 7} \right]$

4 Show, preferably directly via a combinatorial argument that $\left\{ {n \atop k} \right\} \geq \binom{n}{k}$ and decide when there is equality.

5 Prove the identities

$$\left\{ {n+1 \atop m+1} \right\} = \sum_k \binom{n}{k} \left\{ {k \atop m} \right\}$$

and

$$\left[{n+1 \atop m+1} \right] = \sum_k \binom{k}{m} \left[{n \atop k} \right]$$

6 Among the non-zero Stirling numbers $\left[{n \atop k} \right]$ (with $1 \leq k \leq n$ what is the average size?

7 Express $\arctan(z)$ as a power series and then as a hypergeometric series.

8 Let a, b be arbitrary real numbers and consider $G_n = aF_n + bF_{n+1}$ where the F_n are the Fibonacci numbers. Show that G_n satisfies the same recursive formula as the Fibonnaci numbers. In particular find a, b such that $G_n = F_{n+k}$

9 Find the numbers a_n such that we have for each real (complex?) t an identity $(\sum_{n\geq 0} a_n)^t = \sum_{n\geq 0} a_n t^n$

10 Write down the first harmonic number $H_n = \frac{a_n}{b_n}$ where $\gcd(a_n, b_n) = 1$. Show for p primes

i) if $p|b_n$ then $p \leq n$

ii) that there is an infinite number $p \leq n$ such that $p \not| b_n$ or at least show some examples of such primes.

iii) for every p there is an infinite number of n such that $p|b_n$