EXERCISES: CHAPTER 10

Section 10.1 (The addition principle)

1. The rules for the University of Florida five-a-side football competition specify that the members of each team must have birthdays in the same month. How many mathematics students are needed in order to guarantee that they can raise a team ?

2. What is wrong with the following argument ?: Since half of the numbers n in the range $1 \leq n \leq 60$ are multiples of 2, 30 of them cannot be primes. Since one-third of the numbers are multiples of 3, 20 of them cannot be primes. Hence at most 10 of them are primes.

3. Write out a proof (using induction on n) of the fact that, if $A_1,\, A_2,\, \ldots,\, A_n$ are pairwise disjoint finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

4. Show that in any set of 10 people there are either four mutual friends or three mutual strangers.

Section 10.2 (Counting sets of pairs)

1. In Dr. Pepper's calculus class, 32 of the students are boys. Each boy knows five of the girls in the class and each girl knows eight of the boys. How many girls are in the class ?

2. Suppose we have a number of different subsets of $\mathbb{N}_8 = \{1,\, 2,\, \ldots,\, 8\}$, with the property that each one has four elements, and each element of $\mathbb{N}_8$ belongs to exactly three of the subsets. How many subsets are there ? Write down a collection of subsets which satisfies these conditions.

3. Is it possible to find a collection of subsets of $\mathbb{N}_8$ such that each one has exactly three members and each member of $\mathbb{N}_8$ belongs to exactly five of the subsets ?

4. Prove by induction on n that, if $A_1,\, A_2,\, \ldots,\, A_n$ are finite sets then

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|.$$

5. In the language "Simpletonish" there are 26 letters and every word has 4 letters. Every rearrangement of the letters of a word is also a word. How many words are there ? How many words do not contain the letter b ?

Section 10.3 (Euler's function)

1. Find the values of $\phi(19)$, $\phi(20)$, $\phi(21)$.

2. Show that if x and n are coprime, so are $n - x$ and n. Deduce that $\phi(n)$ is even for all $n \geq 3$.

3. Show that, if p is a prime and m is a positive integer, then $\phi(p^m) = p^m - p^{m-1}$.

4. Find a counterexample to the conjecture that $\phi(ab) = \phi(a)\phi(b)$ for all $a,\, b \in \mathbb{N}$. Can you modify the conjecture so that it holds ?

Section 10.4 (Functions, words and selections)

1. How many national flags can be constructed from three equal vertical strips, using the colours red, white, blue and green ? (It is assumed that colours can be repeated and that one vertical edge of the flag is distinguished as the "flagpole side").

2. Write down all the subsets of $\{a,\, b,\, c,\, d\}$ and give an explicit correspondence between them and binary strings of length 4.

3. Keys are made by cutting incisions of various depths in a number of positions on a blank key. If there are 8 possible depths, how many different positions are required to make at least one million different keys ?

4. Show that there are more than 10^{76} subsets of the set of subsets of a set with eight members.

Section 10.5 (Injections as ordered selections without repitition)

1. In how many ways can we select a batting order of 11 from a pool of 14 cricketers ?

2. How many four-letter words can be made from an alphabet of 10 symbols if there are no restrictions on spelling except that no letter can be used more than once ?

3. Explain briefly how you would make a systematic list of all the ordered selections, without repitition, of three things from the set $\{a,\, b,\, c,\, d,\, e,\, f\}$.

4. Show that, if $n > r > m$ are positive integers, then

$$P(n,\, m) \times P(n - m,\, r - m) = P(n,\, r).$$

Review Section 10.7

1. A committee of nine people must elect a chairman, secretary and treasurer. In how many ways can this be done ?

2. In the usual set of dominos each domino may be represented by the symbol $[x \mid y]$, where x and y are members of the set $\{0, 1, 2, 3, 4, 5, 6\}$. The numbers x and y (of dots) may be equal. Explain why there are in total only 28 different dominos, rather than 49.

3. In how many ways can we select a black square and a white square on a chessboard in such a way that the two squares are not in the same rank or the same file ?

4. In how many ways can we place 8 rooks on a chessboard in such a way that no two of them are on the same rank or file ?

5. Suppose there are m boys and n girls in a class. What is the number of ways of arranging them in a line so that all the girls are together ?

6. If we have nine different subsets of $\mathbb{N}_{12}$, each of which has eight members, and each member of $\mathbb{N}_{12}$ occurs in the same number r of subsets, what is the value of r ? Is it possible to find nine different subsets of $\mathbb{N}_{12}$, each of which has seven members, so that each member of $\mathbb{N}_{12}$ occurs in the same number of subsets ?

7. How many five-digit telephone numbers have a digit that occurs more than once ?

10. The rooms of the house in Figure 1 (see attached file) are to be painted in such a way that rooms with an interconnecting door have different colours. If there are n colours available, how many different colour schemes are possible ?

11. Let $X_1 = \{0, 1\}$ and, for $i \geq 2$ define the set X_i to be the set of subsets of X_{i-1}. What is the least i for which $|X_i| > 10^{100}$?

12. Suppose we have a set of generalised dominos in which the numbers (of dots) range from 0 to n. Let k be any integer in the range $0 \leq k \leq n$. Show that the number of dominos $[x \mid y]$ for which $x+y = n-k$ is equal to the number for which $x+y = n+k$.

14. Show that, for any positive integers n and m,

$$\phi(n^m) = n^{m-1}\phi(n).$$

15. Calculate $\phi(1000)$ and $\phi(1001)$.

17. Let u_n denote the number of words of length n in the alphabet $\{0, 1\}$ which have the property that no two zeroes are consecutive. Show that

$$u_1 = 2, \quad u_2 = 3, \quad u_n = u_{n-1} + u_{n-2} \ \forall\, n \geq 3.$$

19. Show that in any set of 20 people there is either a set of 4 mutual friends or a set of 4 mutual strangers.