ANSWERS: CHAPTER 10

Section 10.1

1. At least $12 \times 4 + 1 = 49$ students are needed, by the Extended Pigeonhole Principle.

2. There are 10 numbers divisible by both 2 and 3, and these have been removed twice. In addition, the primes 2 and 3 themselves have been removed erroneously. Hence, the conclusion should be that there are at most 22 primes (there are, in fact, 17 of them - the argument has not yet considered sieving out multiples of 5 and 7).

3. The base case $n = 2$ states that, if $A_1 \cap A_2 = \phi$ then $|A_1 \cup A_2| = |A_1| + |A_2|$. I will simply assert that this is obvious. Suppose the result holds for n sets and consider $n+1$ pairwise disjoint sets $A_1, A_2, \dots, A_{n+1}$. Let $\mathcal{A}_1 := \cup_{i=1}^{n} A_i$ and $\mathcal{A}_2 := A_{n+1}$. Then $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$ so, by the base case, we have $| \cup_{i=1}^{n+1} A_i| = |\mathcal{A}_1 \cup \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2|$. But by the induction hypothesis, $|\mathcal{A}_1| = | \cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$. Putting these two ingredients together yields the result for $n+1$ sets, v.s.v.

4. This is the statement that $R(4, 3) \leq 10$. In class we will prove that $R(3, 3) = 6$, i.e.: in any group of 6 people there must either be three mutual friends or three mutual strangers. Now consider a set S of 10 people. Let p be any one of them. Let S_p be the set of strangers to p. If any two members of S_p are mutual strangers, then these two together with p from a group of three mutual strangers. Hence we may assume that all members of S_p are mutual friends. Thus we are done if $|S_p| \geq 4$, so we may assume $|S_p| \leq 3$ and hence that $|F_p| \geq 6$, where F_p is the set of friends of p. Since $R(3, 3) = 6$ it follows that, inside F_p we can find either three mutual friends or three mutual strangers. In the latter case we are done. In the former case, these three friends together with p form a group of 4 mutual friends. So we're now totally done !

REMARK: The above argument must be totally symmetrical in the concepts "friend" and "stranger". Thus it is also true that, in a group of 10 people, we can always find either 3 mutual friends or 4 mutual strangers. This observation will be used in the solution to Ex. 10.7.19 below.

Section 10.2

1. Consider the set S of ordered pairs (b, g) such that b is a boy and g is a girl who know each other. We are told that, for each b there exist five g:s. Hence $|S| = 32 \times 5 = 160$. We are also told that each girl knows 8 boys, hence $160 = 8G$ where G is the number of girls. Thus, $G = 20$.

2. Let there be k sets. Consider the collection of ordered pairs (s, e), where s is one of the sets and $e \in s$. Since each set has four elements, the number of pairs is $4k$. But since each element of $\mathbb{N}_8$ belongs to three sets, the number of pairs is also $8 \times 3 = 24$. Thus $k = 6$. An example is

$$\{1, 2, 3, 4\}, \ \{1, 3, 5, 7\}, \ \{1, 4, 6, 8\}, \ \{2, 4, 6, 7\}, \ \{2, 5, 7, 8\}, \ \{3, 5, 6, 8\}.$$

3. No. Suppose there are k such sets and again consider the collection of ordered pairs (s, e), where s is one of the sets and $e \in s$. Arguing as in Ex. 2, the number of pairs equals both $3k$ and $5 \times 8 = 40$. This gives $k = \frac{40}{3}$, which is impossible since k must be an integer.

4. Similar to Ex. 10.1.3.

5. By the multiplication principle, there are 26^4 possible words in all, and 25^4 possible words if we can't use b.

Section 10.3

1. 19 is prime so $\phi(19) = 18$. $\phi(20) = \phi(2^2 \cdot 5) = 20 \cdot (1 - \frac{1}{2})(1 - \frac{4}{5}) = 8$. Finally, $\phi(21) = \phi(3 \cdot 7) = 21 \cdot (1 - \frac{1}{3})(1 - \frac{1}{7}) = 12$.

2. The underlying relevant fact here is that, if $d|a$ and $d|b$, then $d|(a \pm b)$. Suppose $n - x$ and n have a common factor d. Then $d|n - (n - x)$, i.e.: $d|x$, so d is also a common factor of x and n. And vice versa by a similar argument. Thus GCD$(x, n) = 1$ if and only if GCD$(n - x, n) = 1$.

Now let $n \geq 3$. If n is odd then $n/2$ is not an integer, whereas if n is even then $n/2$ is an integer greater than one, thus GCD$(n/2, n) = n/2 > 1$ in this case. It follows that the numbers between 1 and n which are relatively prime to n can, by the previous paragraph, be grouped into disjoint pairs $\{x, n - x\}$. In particular, there are an even number of such numbers, v.s.v.

3. The numbers between 1 and p^m which have a common factor with p^m are precisely those which are multiples of p, since p is prime. There are $p^m/p = p^{m-1}$ such multiples of p. Thus $\phi(p^m) = p^m - p^{m-1}$, v.s.v.

4. For example, take $a = 4$, $b = 6$. We have $\phi(4) = \phi(6) = 2$ so $\phi(4)\phi(6) = 4$. But $\phi(24) = \phi(2^3 \cdot 3) = 24 \cdot (1 - \frac{1}{2})(1 - \frac{1}{3}) = 8$.

The conjecture holds if and only if GCD$(a, b) = 1$, in all other cases $\phi(ab) > \phi(a)\phi(b)$. Indeed this follows immediately from formula (3.3) in the lecture notes. One can also deduce it from the so-called *Chinese Remainder Theorem* (see wiki if you're interested).

Section 10.4

1. $4^3 = 64$ by the multiplication principle. Note by the way that in any "sensible" flag, the same colour won't be used on two adjacent strips. If we discount those, then there are only 36 possible flags since: (i) there are 4 flags with the same colour on all three strips (ii) there are $4 \times 3 \times 2 = 24$ flags with the same colour on two adjacent strips and a different colour on the third strip - 4 possibilities for the first colour, 3 for the second colour and 2 for the stripe to hold the second colour (it can't be the middle stripe).

2. For example,

$$\begin{aligned}\phi &\leftrightarrow 0000,\\ \{d\} \leftrightarrow 0001, \quad \{c\} \leftrightarrow 0010, \quad \{b\} \leftrightarrow 0100, \quad \{a\} &\leftrightarrow 1000,\\ \{c,\, d\} \leftrightarrow 0011, \quad \{b,\, d\} \leftrightarrow 0101, \quad \{a,\, d\} &\leftrightarrow 1001,\\ \{b,\, c\} \leftrightarrow 0110, \quad \{a,\, c\} \leftrightarrow 1010, \quad \{a,\, b\} &\leftrightarrow 1100,\\ \{b,\, c,\, d\} \leftrightarrow 0111, \quad \{a,\, c,\, d\} \leftrightarrow 1011, \quad \{a,\, b,\, d\} \leftrightarrow 1101, \quad \{a,\, b,\, c\} &\leftrightarrow 1110,\\ \{a,\, b,\, c,\, d\} &\leftrightarrow 1111.\end{aligned}$$

3. We seek the smallest integer k such that $8^k > 10^6$. The smallest power of 2 that is greater than 10^6 is $1024^2 = (2^{10})^2 = 2^{20} = 8^{20/3}$. Thus $k = 7$.

4. There are 2^8 subsets of a set with eight elements and then 2^{2^8} subsets of this set. So we need to check that $2^{2^8} = 2^{256} > 10^{76}$. But $1024 = 2^{10} > 10^3$, so $2 > 10^{0.3}$ and thus $2^{256} > 10^{256\times 0.3} = 10^{76.8} > 10^{76}$, v.s.v.

Section 10.5

1. The order matters, so $P(14,\, 11) = \frac{14!}{3!}$.

2. $P(10,\, 4) = 10 \times 9 \times 8 \times 7 = 5040$.

3. There are $P(6,\, 3) = 6 \times 5 \times 4 = 120$ such selections. I'd list them in alphabetical order.

4. The right-hand side is the number of ways to choose r distinct objects from n, when the order matters. Since the order matters, each choice of r objects can be broken down into a pair of choices: first, an ordered choice of m objects; second, an ordered choice of $r - m$ from the remaining $n - m$ objects. There are $P(n,\, m)$ possibilities for the first choice and $P(n - m,\, r - m)$ for the second, hence $P(n,\, m) \times P(n - m,\, r - m)$ for the pair of choices, by the multiplication principle.

Review Section 10.7

1. The order matters since these are three different jobs. Thus $P(9,\,3) = 9 \times 8 \times 7 = 504$.

2. Because you can turn the domino 180 degrees, i.e.: $[x\,|\,y]$ is the same domino as $[y\,|\,x]$. There are thus only $(7 \times 6)/2 = 21$ different dominoes where $x \neq y$. There are 7 where $x = y$. Thus 28 in all.

3. There are 32 choices for the black square. This rules out 8 white squares (4 on the same rank and 4 on the same file), leaving 24 choices for the latter. The order matters, so there are $32 \times 24 = 768$ choices for the pair of squares.

4. $8!$, since one must place one rook on each rank and then each placement corresponds to a permutation of the files (or vice versa).

5. There are $m+n$ positions in the line and the "leftmost" girl can stand in any of positions 1 through $m+1$. The n girls can then be lined up in $n!$ ways, and the boys in $m!$ ways. By MP, the total number of possible arrangements is $(m+1) \times n! \times m! = n! \times (m+1)!$.

6. Similar reasoning to Exs. 10.2.2 and 10.2.3. In the first part, $r = 6$. In the second part, we'd be led to $r = \frac{9\times 7}{12} = \frac{21}{4} \notin \mathbb{Z}$, so this is impossible.

7. There are 10^5 five-digit telephone numbers in all. There are $P(10,\,5) = 10 \times 9 \times 8 \times 7 \times 6 = 30240$ without any repeated digits. Hence, there are $10^5 - 30240 = 69760$ numbers with some digit repeated.

10. The rooms can be denoted $1,\,2,\,3,\,4$ such that $1,\,2,\,3$ are all connected together, plus that 3 is connected to 4. Rooms $1,\,2,\,3$ must get three different colours so there are $P(n,\,3) = n(n-1)(n-2)$ ways to colour these three. Then room 4 can get any colour other than that given to room 3, so there are $n-1$ choices for that room. By MP, the total number of possible colourings is $n(n-1)^2(n-2)$.

11. $|X_i| = 2^{|X_{i-1}|}$. Thinking fuzzily, $10^3 \approx 2^{10}$ so $10^{100} \approx 2^{333}$. We have $|X_2| = 2^2 = 4$, $|X_3| = 2^4 = 16$, $|X_4| = 2^{16} > 333$, so $|X_5| > 10^{100}$. The answer is $i = 5$.

12. The correspondence $[x\,|\,y] \leftrightarrow [n-x\,|\,n-y]$ gives a 1-1 correspondence between dominos with sum $x+y = n-k$ and those with sum $(n-x)+(n-y) = 2n-(x+y) = 2n-(n-k) = n+k$, v.s.v.

14. Follows immediately from formula (3.3) in the lecture notes, since the primes dividing n^m are, for any m, precisely those dividing n itself.

15. $\phi(1000) = \phi(2^3 \cdot 5^3) = 1000 \cdot (1-\frac{1}{2})(1-\frac{1}{5}) = 400$.

$\phi(1001) = \phi(7 \cdot 11 \cdot 13) = 1001 \times \frac{6}{7} \times \frac{10}{11} \times \frac{12}{13} = 6 \times 10 \times 12 = 720.$

17. Clearly $u_1 = 2$ since there are two possible words, namely 0 and 1. Similarly, $u_2 = 3$ since the possibilities are 01, 10 and 11. Now let $n \geq 3$ and consider an admissable word of length n.

CASE 1: The first digit is 0. Then the second digit must be 1. We can choose the remaining $n-2$ digits freely, except that we have the same restriction as at the outset - thus there are u_{n-2} posssible words.

CASE 2: The first digit is 1. This time we can choose the remaining $n-1$ digits freely, but for having the same restriction as at the outset - thus there are u_{n-1} possible words.

By the addition principle, there are $u_{n-2} + u_{n-1}$ possible words of length n, i.e.: $u_n = u_{n-2} + u_{n-1}$, v.s.v.

19. Let p be any person. There are 19 others so, by the Pigeonhole Principle, either p has at least 10 friends or at least 10 strangers.

CASE 1: p has at least 10 strangers. By Ex. 10.1.4, amongst p's strangers we can find either 4 mutual friends or 3 mutual strangers. In the former case we are done. In the latter case, any such 3 mutual strangers together with p form a group of 4 mutual strangers, and we are done again.

CASE 2: p has at least 10 friends. By the remark following our solution to Ex. 10.1.4, amongst p's friends we can find either 3 mutual friends or 4 mutual strangers. In the latter case we are done. In the former case, any such 3 mutual friends together with p form a group of 4 mutual friends, and we are done again.