EXERCISES: CHAPTER 11

Section 11.1 (Binomial numbers)

1. Show that $\binom{n}{r} = 0$ if $r > n$.

2. Find the values of $\binom{n}{0}$, $\binom{n}{1}$ and $\binom{n}{n}$ for all $n \geq 1$.

3. Prove that $\binom{n}{r} = \binom{n}{n-r}$ for all $0 \leq r \leq n$.

4. Given that the fifth row[1] of Pascal's triangle is $1,\ 5,\ 10,\ 10,\ 5,\ 1$, write down the sixth through eighth rows.

5. Evaluate $\binom{16}{4}$ and $\binom{17}{5}$.

6. Explain why the number of words of length n in the alphabet $\{0,\ 1\}$ which contain exactly r zeroes is $\binom{n}{r}$.

7. Let $s,\ n$ be positive integers. Prove the identity

$$\binom{s-1}{0} + \binom{s}{1} + \cdots + \binom{s+n-2}{n-1} + \binom{s+n-1}{n} = \binom{s+n}{n}.$$

Section 11.2 (Unordered selections with repitition)

1. Write down the formulas for the number of ways to place r balls into n distinguishable bins when (i) the balls are identical (ii) the balls are distinguishable (iii) the balls are identical and no bin can receive more than one ball (iv) the balls are distinguishable and no bin can receive more than one ball. Compute all four quantities for $n = 5$, $r = 3$.

2. Show that when three identical dice are thrown there are 56 possible outcomes. What is the number of possible outcomes when n identical dice are thrown ?

3. Suppose that the expression $(x + y + z)^n$ is expanded and terms collected in the usual manner. How many terms are in the resulting expression ? E.g.: the expansion

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

contains 6 terms.

[1] Here I adopt the convention that a single"1" constitutes the *zeroth* row of the triangle, corresponding to $(x + y)^0 = 1$ in the Binomial Theorem.

Section 11.3 (The binomial theorem)

1. Write out the formulae for $(1+x)^8$ and $(1-x)^8$.

2. Calculate the coefficients of

(i) x^5 in $(1+x)^{11}$
(ii) a^2b^8 in $(a+b)^{10}$
(iii) a^6b^6 in $(a^2+b^3)^5$
(iv) x^3 in $(3+4x)^6$.

3. Use the identity $(1+x)^m(1+x)^n = (1+x)^{m+n}$ to prove that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0},$$

where m, n, r are positive integers such that $r \leq \min\{m, n\}$. Can you give an alternative proof by "combinatorial reasoning" ?

4. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Can you give and prove a similar formula involving 3^n ?

5. Show that if r and s are integers such that $s|r$, and p is a prime dividing r but not s, then p divides r/s. Deduce that
(i) the binomial number $\binom{p}{i}$ is divisible by p for all values of i in the range $1 \leq i \leq p-1$;
(ii) $(a+b)^p - a^p - b^p$ is divisible by p for all integers a and b.

Section 11.4 (The sieve principle)

1. In a class of 67 math students, 47 can read French, 35 can read German and 23 can read both. How many can read neither ? Suppose, furthermore, that 20 can read Russian and, of these, 12 can also read French, 11 can also read German and 5 can read all three languages. How many cannot read any of the three languages ?

2. Find the number of ways of arranging the letters A, E, M, O, U, Y in a sequence in such a way that neither of the words ME and YOU occurs.

3. Caclulate the number d_4 of derangements of $\{1, 2, 3, 4\}$ and make a full list of them.

4. Use the formula for d_n derived in class and induction to prove the recursive formulas

$$d_1 = 0, \quad d_2 = 1, \quad d_n = (n-1)(d_{n-1} + d_{n-2}) \ \forall\, n \geq 3.$$

5. Can you prove the above recursion instead by a "combinatorial reasoning" ?

Section 11.5 (Some arithmetical applications)

Some of the exercises in this section involve the following two notions:

(a) The so-called *Möbius function* $\mu : \mathbb{N} \to \mathbb{N}$ which is defined as follows:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } p^2 | n \text{ for some prime } p; \\ (-1)^k, & \text{if } n = p_1 \times p_2 \times \cdots \times p_k, \text{ for } k \textit{ distinct} \text{ primes.} \end{cases}$$

(b) A *divisor* of a positive integer n means another positive integer d such that $d|n$ ($d = 1$ and $d = n$ are allowed !).

1. Calculate $\phi(n)$ and $\mu(n)$ for each n in the range $95 \leq n \leq 100$.

2. Show that if the prime factorisation of n is $p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ then the number of divisors of n is $(e_1 + 1)(e_2 + 1) \dots (e_r + 1)$.

3. Prove that if GCD$(m, n) = 1$ then $\phi(mn) = \phi(m)\phi(n)$.

4. Show that if $1 \leq x \leq n$ then GCD$(x, n) =$ GCD$(n - x, n)$. Deduce that, if $n \geq 2$, then the sum of all integers x satisfying $1 \leq x \leq n$ and GCD$(x, n) = 1$ is $\frac{1}{2} n\phi(n)$.

5. Prove that, for all $n \in \mathbb{N}$,

$$\phi(n) = \sum_{d|n} \mu(n) \frac{n}{d}.$$

6. Prove that, for all $n \in \mathbb{N}$,

$$\sum_{d|n} \phi(d) = n.$$

Review Section 11.8

1. Write out the formulae for $(x + y)^9$ and $(x - y)^9$.

2. Caclulate the coefficient of

(a) x^6 in $(1 + x)^{12}$
(b) $a^3 b^7$ in $(a + b)^{10}$
(c) $a^4 b^6$ in $(a^2 + b)^8$.

3. Prove that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

4. Suppose that n points are drawn on a circle in such a way that no three of the diagonals connecting pairs of points are concurrent. How many internal points of intersection are there between diagonals ?

6. Show that when $n \geq m$

$$\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}.$$

7. Let X be a set with n elements. Show that

(i) there is a collection of $\binom{n-1}{k-1}$ k-element subsets of X each pair of which has a non-empty intersection;

(ii) there is a collection of $\binom{n}{m}$ subsets of X with the property that not one of them is contained in any other, where $m = \lfloor n/2 \rfloor$.

13. How many integers x in the range $1 \leq x \leq 100$ are not divisible by 2, 3 or 5 ?

14. Professor McBrain has taught the same course for the last 12 years and tells three jokes each year. He has never told the same set of three jokes twice (the order of the jokes is unimportant). How many jokes must he know (at least) ?

16. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be *multiplicative* if $f(mn) = f(m)f(n)$ whenever GCD$(m, n) = 1$. Show that if f is multiplicative then so is the function g defined by

$$g(n) = \sum_{d|n} f(d).$$

17. Write down formulae for the multiplicative functions

$$\text{(i)} \sum_{d|n} \mu(d)\phi(d), \qquad \text{(ii)} \sum_{d|n} \frac{\mu(d)}{\phi(d)}$$

in terms of the prime factorisation of n.

18. Let $\sigma_k(n)$ denote the sum of the kth powers d^k, taken over all divisors d of n. Show that σ_k is a multiplicative function for each k and write a formula for it in terms of the prime factorisation of n.

19. Let $S_r(n)$ be the sum of the rth powers of the first n positive integers. Show that for $r \geq 1$

$$(n+1)^{r+1} - (n+1) = \sum_{i=1}^{r} \binom{r+1}{i} S_{r-i+1}(n).$$

Deduce that there is a formula for $S_r(n)$ which is a polynomial of degree $r+1$ in n.

20. Give a proof of the binomial theorem using induction and Pascal's identity.