ANSWERS: CHAPTER 11

Section 11.1

1. There are obviously zero ways to choose r distinct objects from n if $r > n$. By the way, note that the formula (1.9) will also give zero when $r > n$ since then one of the factors in the numerator will be zero.

2. $\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = n$.

3. Proposition 1.5 in the lecture notes.

4.

$$\begin{array}{ccccccccccccccccc}
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

5.

$$\binom{16}{4} = \frac{16 \times 15 \times 14 \times 13}{1 \times 2 \times 3 \times 4} = 4 \times 5 \times 7 \times 13 = 1820,$$

$$\binom{17}{5} = \binom{16}{4} + \binom{16}{5} = \binom{16}{4} + \binom{16}{4} \times \frac{12}{5} = \binom{16}{4} \times \frac{17}{5} = \frac{1820 \times 17}{5} = 6188.$$

6. Since specifying a word involves choosing which r of the n digits are to be zeroes.

7. This can be proven by repeated use of Pascal's identity (2.2). The underlying combinatorial reasoning is as follows. The RHS is the number of ways to choose n distinct numbers from the set $\{1, 2, \ldots, s+n\}$. The largest number not chosen must be some i such that $s \leq i \leq n$ (as otherwise more than n numbers would have been chosen in all). Fix such an i and let $i = s + j$, for some $0 \leq j \leq n$. In this case, each of the numbers $s+j+1, \ldots, s+n$ is chosen, leaving j numbers to be chosen from among $1, 2, \ldots, s-1+j$. There are $\binom{s-1+j}{j}$ ways in which this can be done. Finally, the addition principle implies that the total number of ways to choose n numbers from $s+n$ is $\sum_{j=0}^{n} \binom{s-1+j}{j}$, which is the LHS.

Section 11.2

1.

$$\text{(i)} \ \binom{r+n-1}{r}, \quad \text{(ii)} \ n^r, \quad \text{(iii)} \ \binom{n}{r}, \quad \text{(iv)} \ P(n, r).$$

For $n = 5$, $r = 3$, the corresponding numbers are 35, 125, 10, 60.

2. Let x_i denote the number of dice showing i, where $1 \le i \le 6$. Since the dice are identical, all that matters is how many dice show each number. Thus we wish to count the number of solutions in non-negative integers to $x_1 + \cdots + x_6 = 3$ which, by Example 2.11, is $\binom{3+6-1}{3} = \binom{8}{3} = 56$.

When n dice are thrown, we are counting solutions to $x_1 + \cdots + x_6 = n$, so there are $\binom{n+6-1}{n} = \binom{n+5}{5} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120}$ possible outcomes.

3. A typical term is $x^{a_1}y^{a_2}z^{a_3}$ where $a_1 + a_2 + a_3 = n$ and $a_i \in \mathbb{N}_0$. Thus the number of terms is $\binom{n+3-1}{n} = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}$.

Section 11.3

1. The coefficients have already been worked out in Ex. 11.1.4.

$$(1+x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8,$$
$$(1-x)^8 = 1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8.$$

2.

(i) $\binom{11}{5} = 462$, (ii) $\binom{10}{2} = 45$, (iii) $\binom{5}{3} = 10$, (iv) $\binom{6}{3} \times 3^3 \times 4^3 = 34560$.

3. The LHS is the coefficient of x^r in $(1+x)^{m+n}$, hence is also the coefficient of x^r in $(1+x)^m(1+x)^n$. We get a term x^r in the latter by multplying $x^a x^b$ for some $0 \le a \le m$, $0 \le b \le n$ and $a + b = r$. Thus the contribution to x^r, for a fixed pair $(a, r-a)$, is $\binom{m}{a}\binom{n}{r-a}$. So the total contribution is $\sum_{a=0}^{r} \binom{m}{a}\binom{n}{r-a}$, v.s.v.

Combinatorially, the LHS is the number of ways to choose r distinct numbers from $\{1, 2, \ldots, m+n\}$. On the RHS, the term $\binom{m}{a}\binom{n}{r-a}$ correspopnds to choosing a of the r numbers from among $\{1, 2, \ldots, m\}$ and the remaining $r - a$ from among $\{m+1, m+2, \ldots, m+n\}$.

4. The first identity is obtained by applying the Binomial Theorem (2.1) with $x = y = 1$. One can also reason "combinatorially": the LHS evidently counts the total number of subsets of an n-element set (addition principle) and so does the RHS since, when choosing a subset, one has two choices for each element - whether to include it or not (mutliplication principle).

If one instead applies the binomial theorem with $x = 1$, $y = 2$ one obtains the formula $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 3^n$. Here the RHS counts the number of ternary (base-3) strings of length n. On the LHS, the kth term counts the number of such strings with k zeroes, since we first choose where to put the zeroes and then have two choices for each of the remaining $n - k$ digits.

By the way, note that if we instead put $x = 2$, $y = 1$ then we obtain the slightly nicer formula $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$. Moreover, the first formula can be rewritten as $\sum_{k=0}^{n} \frac{1}{2^k}\binom{n}{k} = \left(\frac{3}{2}\right)^n$.

5. The first assertion follows immediately from FTA.

(i) $\binom{p}{i} = \frac{p(p-1)\ldots(p-i+1)}{1\times 2\times\cdots\times i}$. A priori, we know this is an integer since it is the number of ways to choose i objects from p. In the formula, the numerator is divisible by p whereas the denominator isn't (since p is prime). Hence, the whole expression must be an integer divisible by p, by FTA.

(ii) Using the binomial theorem, we obtain

$$(a+b)^p - a^p - b^p = \sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i}.$$

As shown in part (i), each coefficient is divisible by p, hence so is the entire expression whenever a, b are integers.

Section 11.4

1. Let X be the set of all students in the class and let A, B, C be the subsets consisting of those who read French, German and Russian respectively.

First we are told that $|X| = 67$, $|A| = 47$, $|B| = 35$ and $|A \cap B| = 23$. It follows that

$$|X \backslash (A \cup B)| = |X| - |A \cup B| = |X| - |A| - |B| + |A \cap B| = 67 - 47 - 35 + 23 = 8.$$

Thus, 8 students speak neither French nor German.

Second, we are told that $|C| = 20$, $|A \cap C| = 12$, $|B \cap C| = 11$ and $|A \cap B \cap C| = 5$. It follows that

$$\begin{aligned} |X \backslash (A \cup B \cup C)| &= |X| - |A \cup B \cup C| = \\ = |X| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| &= \\ = 67 - (47 + 35 + 20) + (23 + 12 + 11) - 5 &= 6. \end{aligned}$$

Thus, 6 students speak none of the three languages.

2. Let X denote the set of all possible arrangements of the six letters and let A (resp. B) denote the subset consisting of those arrangements in which "ME" (resp. "YOU") occurs. We seek $|X \backslash (A \cup B)| = |X| - |A| - |B| + |A \cap B|$.

Firstly, $|X| = 6! = 720$. Secondly, $|A| = 5! = 120$ since we may consider "ME" as one "letter" and thus we can permute freely 5 letters instead of 6. Similarly, $|B| = 4! = 24$ and $|A \cap B| = 3! = 6$. Thus, the number of admissable words is $720 - 120 - 24 + 6 = 582$.

3. From (3.10) in the notes, we have

$$d_4 = 4! \times \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 24\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 24 \times \frac{9}{24} = 9.$$

A full list is $(1,\,2,\,3,\,4) \mapsto (a,\,b,\,c,\,d)$ where $(a,\,b,\,c,\,d)$ is one of the following:

$$\begin{gathered}(2,\,1,\,4,\,3),\quad (2,\,3,\,4,\,1),\quad (2,\,4,\,1,\,3),\quad (3,\,1,\,4,\,2),\quad (3,\,4,\,1,\,2),\\ (3,\,4,\,2,\,1),\quad (4,\,1,\,2,\,3),\quad (4,\,3,\,1,\,2),\quad (4,\,3,\,2,\,1).\end{gathered}$$

4. It is obvious that $d_1 = 0$ and $d_2 = 1$. The recursion can be rewritten as

$$d_n + d_{n-1} = nd_{n-1} + (n-1)d_{n-2}. \tag{0.1}$$

Recall from the lecture notes the formula

$$d_m = m! \times \left(\sum_{k=0}^{m} \frac{(-1)^k}{k!} \right).$$

Substitute this into the RHS of (0.1) and note that $n! = n \times (n-1)!$, $(n-1)! = (n-1) \times (n-2)!$. This gives

$$nd_{n-1} + (n-1)d_{n-2} = n! \times \left(\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) + (n-1)! \times \left(\sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \right). \tag{0.2}$$

The first term on the RHS of (0.2) is just $d_n - n! \times \frac{(-1)^n}{n!} = d_n - (-1)^n$. The second term is $d_{n-1} - (n-1)! \times \frac{(-1)^{n-1}}{(n-1)!} = d_{n-1} - (-1)^{n-1}$. Hence their sum is $(d_n + d_{n-1}) - ((-1)^n + (-1)^{n-1}) = (d_n + d_{n-1}) - 0 = d_n + d_{n-1}$, v.s.v.

5. Consider a derangement π of $\{1, \ldots, n\}$. In particular $\pi(1) \neq 1$, so there are $n-1$ choices for $\pi(1)$. It thus suffices to show that, for each fixed choice of $i = \pi(1)$, there are $d_{n-1} + d_{n-2}$ possible derangements. Fix i and consider two cases:

CASE 1: $\pi(i) = 1$. There are $n-2$ numbers left to permute and π must act as a derangment on these, hence there are d_{n-2} possibilites for π.

CASE 2: $\pi(i) \neq 1$. We can imagine this condition as saying that π acts as a derangement on the $n-1$ "place holders" $2, 3, \ldots, n$, where we think of place i as holding the number 1. Thus there are d_{n-1} possibilities for π.

Finally, apply the addition principle.

Section 11.5

1. From formula (3.3), the definition of μ and the prime factorisations $95 = 5 \cdot 19$, $96 = 2^5 \cdot 3$, $97 = 97$, $98 = 2 \cdot 7^2$, $99 = 3^2 \cdot 11$, $100 = 2^2 \cdot 5^2$, we easily compute the values

$$\phi(95) = 72, \ \ \phi(96) = 32, \ \ \phi(97) = 96, \ \ \phi(98) = 42, \ \ \phi(99) = 60, \ \ \phi(100) = 40,$$
$$\mu(95) = 1, \ \ \mu(96) = 0, \ \ \mu(97) = -1, \ \ \mu(98) = \mu(99) = \mu(100) = 0.$$

2. By FTA, every divisor of n has the form $p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$, where $0 \leq a_i \leq e_i$ for each i. There are thus $e_i + 1$ choices for a_i and hence a total of $\prod_{i=1}^{r}(e_i + 1)$ divisors, by MP.

3. See the solution to Ex. 10.3.4.

4. For the first assertion, see the solution to Ex. 10.3.2. As shown there, it follows that, if $n \geq 3$, then the numbers x between 1 and n satisfying GCD$(x, n) = 1$ can be grouped into $\frac{\phi(n)}{2}$ pairs $\{x, n-x\}$. The sum of each pair is obviously n, hence the sum over all x is $\left(\frac{\phi(n)}{2}\right) n$, v.s.v. This leaves just $n = 2$ to check. In this case,

$\frac{1}{2}n\phi(n) = \frac{1}{2} \times 2 \times 1 = 1$ and, since 1 is the only number $x \in \{1, 2\}$ which satisfies $\text{GCD}(x, 2) = 1$, the formula holds even in this case.

5. If you think about how the Möbius function is defined, you'll see that this is just a recasting of formula (3.3).

6. STEP 1: The function $f(n) = n$ is obviously multiplicative (see Ex. 11.8.16 for the definition). I claim this is also the case for the function $g(n) = \sum_{d|n} \phi(d)$. Indeed, by Ex. 11.5.3, this is just a special case of Ex. 11.8.16.

STEP 2: A multiplicative function is determined completely by its values at prime powers since

$$f\left(p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}\right) = f(p_1^{e_1}) f(p_2^{e_2}) \dots f(p_r^{e_r}).$$

Hence, in order to show that two multiplicative functions $f(n)$ and $g(n)$ are equal for all $n \in \mathbb{N}$, it suffices to prove this when n is a prime power. So let $n = p^k$. Then the divisors of p^k are just p^j, where $0 \leq j \leq k$. Moreover, $\phi(p^0) = 1$ and, for each $j \geq 1$, $\phi(p^j) = p^j - p^{j-1}$. Hence,

$$\sum_{d|p^k} \phi(d) = 1 + \sum_{j=1}^{k} (p^j - p^{j-1}).$$

This is a telescoping sum (cancellation of terms in pairs) and thus becomes simply p^k, v.s.v.

Review Section 11.8

1.

$$(x+y)^9 = x^9 + 9x^8y + 36x^7y^2 + 84x^6y^3 + 126x^5y^4 + 126x^4y^5 + 84x^3y^6 + 36x^2y^7 + 9xy^8 + y^9,$$
$$(x-y)^9 = x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9.$$

2.

$$\text{(a)} \binom{12}{6} = 2772, \quad \text{(b)} \binom{10}{3} = 120, \quad \text{(c)} \binom{8}{2} = 28.$$

3. Both sides count the number of pairs (A, B), where A and B are subsets of $\{1, 2, \dots, n\}$ satisfying $|A| = r$, $|B| = k$, $B \subseteq A$. On the left, one first chooses A in one of $\binom{n}{r}$ possible ways and then chooses B from A in one of $\binom{r}{k}$ possible ways. On the right, one first chooses B in one of $\binom{n}{k}$ possible ways, and then fills out A in one of $\binom{n-k}{r-k}$ possible ways.

4. Imagine numbering the points clockwise as $1, 2, \dots, n$. Given any 4-tuple $i < j < k < l$ of points, there will be an internal intersection point between the diagonals connecting the pairs $\{i, k\}$ and $\{j, l\}$, but not between any other pair of diagonals formed from these four points. Hence the number of internal intesection points is just the number of 4-tuples (since no three diagonals are concurrent, so no point is repeated), i.e.: simply $\binom{n}{4}$.

6. The right-hand side counts the number of ways to choose $m+1$ distinct numbers from $\{1, 2, \ldots, n+1\}$. Let i be the largest number chosen. Thus $m+1 \le i \le n+1$. Given i, one has to choose m of the numbers $1, 2, \ldots, i-1$, which can be done in $\binom{i-1}{m}$ ways. Thus, the total number of possibilities is $\sum_{i=m+1}^{n+1} \binom{i-1}{m}$, which is just the LHS.

7. WLOG, let $X = \{1, 2, \ldots, n\}$. In (i), we can take all the subsets containing 1. In (ii), we can take all the subsets of size $\lfloor n/2 \rfloor$.

Remark: There are theorems saying that one cannot find larger collections of subsets satisfying the conditions in (i) and (ii), but these are harder to prove. For (i), the relevant result is called the *Erdős-Ko-Rado theorem*. For (ii), the result is called *Sperner's theorem*. See wiki if you're interested.

13. The number of such integers is $|X \backslash (A_2 \cup A_3 \cup A_5)|$, where $X = \{1, 2, \ldots, 100\}$ and A_i is the subset of X consisting of those numbers which are divisible by i. By Inclusion-Exclusion, the number we seek is

$$|X| - |A_2| - |A_3| - |A_5| + |A_2 \cap A_3| + |A_2 \cap A_5| + |A_3 \cap A_5| - |A_2 \cap A_3 \cap A_5| =$$
$$= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

14. He must know at least n jokes, where n is the smallest integer satisfying $\binom{n}{3} \ge 12$. Thus $n = 6$.

16. Let GCD$(m, n) = 1$ and note that, by FTA, the divisors of mn are precisely the numbers $d_1 d_2$ such that $d_1 | m$ and $d_2 | n$. Moreover, GCD$(d_1, d_2) = 1$ for any such pair (d_1, d_2). Thus

$$g(mn) = \sum_{d|mn} f(d) = \sum_{d_1|m} \sum_{d_2|n} f(d_1 d_2) \overset{f \text{ is mult.}}{=} \sum_{d_1|m} \sum_{d_2|n} f(d_1) f(d_2) = \text{(stare)} =$$
$$= \left(\sum_{d_1|m} f(d_1) \right) \left(\sum_{d_2|n} f(d_2) \right) = g(m) g(n), \quad \text{v.s.v.}$$

17. Note that, if p is a prime, then $\mu(1) = 1$, $\mu(p) = -1$ and $\mu(p^j) = 0$ for all $j \ge 2$, while $\phi(1) = 1$ and $\phi(p) = p - 1$. Hence, if $n = p^k$ then

$$\sum_{d|n} \mu(d)\phi(d) = \begin{cases} 1, & \text{if } k = 0, \\ 2 - p, & \text{if } k \ge 1, \end{cases} \quad \text{and} \quad \sum_{d|n} \frac{\mu(d)}{\phi(d)} = \begin{cases} 1, & \text{if } k = 0, \\ \frac{p-2}{p-1}, & \text{if } k \ge 1. \end{cases}$$

Hence if $n \ge 2$ and has prime factorisation $n = \prod_{i=1}^{r} p_i^{e_i}$, then

$$\sum_{d|n} \mu(d)\phi(d) = \prod_{i=1}^{r} (2 - p_i) \quad \text{and} \quad \sum_{d|n} \frac{\mu(d)}{\phi(d)} = \prod_{i=1}^{r} \frac{p_i - 2}{p_i - 1}.$$

18. To show that σ_k is multiplicative can be done in a similar way to Ex. 11.8.16. If $n = p^e$ then $\sigma_k(n) = \sum_{f=0}^{e} p^{kf} = \frac{p^{k(e+1)} - 1}{p^k - 1}$. Thus if $n = \prod_{i=1}^{r} p_i^{e_i}$ then $\sigma_k(n) = \prod_{i=1}^{r} \frac{p_i^{k(e_i+1)} - 1}{p_i^k - 1}$.

19. The identity can be proven combinatorially, i.e.: both sides count something, but in different ways. See Homework 1.

20. Theorem 2.1 is obviously true for $n = 1$. Suppose $n \geq 1$ and it is true that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{0.3}$$

We wish to deduce that

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}. \tag{0.4}$$

But $(x+y)^{n+1} = (x+y)(x+y)^n$. To get a term $x^k y^{n+1-k}$ we either choose x from the first factor and $x^{k-1}y^{n-(k-1)}$ from the second, or choose y from the first factor and $x^k y^{n-k}$ from the second. From (0.3) it follows that the total coefficient of $x^k y^{n+1-k}$ in $(x+y)^{n+1}$ will be $\binom{n}{k-1} + \binom{n}{k}$. But this sum is just $\binom{n+1}{k}$, by Pascal's identity, which proves (0.4).