EXERCISES: CHAPTER 12

Section 12.1 (Partitions of a set)

1. Given that the seventh row of the "Stirling triangle" is $1,\ 63,\ 301,\ 350,\ 140,\ 21,\ 1$, compute the eighth row.

2. Give direct (i.e.: without recourse to the recursion formula) proofs of the identities

$$S(n,\,2) = 2^{n-1} - 1, \quad S(n,\,n-1) = \binom{n}{2}.$$

3. Suppose we are given a partition of the n-set X into k parts and we delete the part containing a given element x. We thereby obtain a partition of a subset Y of X into $k-1$ parts, where $r = |Y|$ is in the range $0 \leq r \leq n-1$. Use this idea to prove that

$$S(n,\,k) = \sum_{r=0}^{n-1} \binom{n-1}{r} S(r,\,k-1).$$

Section 12.2 (Classification and equivalence relations)

1. Let $X = \{1,\,2,\,5,\,6,\,7,\,9,\,11\}$ and define $x \sim x'$ to mean that $x - x'$ is divisible by 5. Verify that $\sim$ is an equivalence relation and describe the partition of X into equivalence classes.

2. Suppose that four chairs labelled $1,\,2,\,3,\,4$ are arranged round a circular table with equal spaces between them. A *seating plan* for four people $A,\,B,\,C,\,D$ can be described by an array such as

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ C & A & B & D \end{array}$$

which means that C occupies chair 1, A occupies chair 2 and so on. Two seating plans are *related* (in the relation $\mathcal{R}$) if one of them can be obtained from the other by moving everyone the same number of places to the right.

(i) What is the total number of seating plans ?
(ii) Show that $\mathcal{R}$ is an equivalence relation.
(iii) Determine the number of equivalence classes and give a representative for each one.

3. With terminology as in Ex. 2, find the number of seating plans and the number of equivalence classes when there are n people and n chairs.

4. Define a relation $\approx$ on $\mathbb{Z}$ by the rule that $n \approx n'$ iff $nn' > 0$. Determine which of the three properties reflexivity, symmetry and transitivity this relation possesses.

5. What is wrong with the following "proof" that symmetry + transitivity $\Rightarrow$ reflexivity:
- If $a \sim b$ then $b \sim a$, by symmetry
- But $a \sim b$ and $b \sim a$ imply $a \sim a$, by transitivity. Hence $a \sim a$ for all a.

Section 12.3 (Distributions and multinomial numbers)

1. How many words can be made from the letters of the word MISSISSIPPI ?

2. Evaluate the multinomial numbers $\binom{10}{4,\,3,\,2,\,1}$ and $\binom{9}{5,\,2,\,2}$.

3. Show that the number of possible positions after four moves of a game of tic-tac-toe is 756.

4. Show that if $a + b + c = n$ then
$$\binom{n}{a,\, b,\, c} = \binom{n-1}{a-1,\, b,\, c} + \binom{n-1}{a,\, b-1,\, c} + \binom{n-1}{a,\, b,\, c-1}.$$
Write down the analogous formula for a general multinomial number.

5. Calculate the coefficient of

(i) $x^5y^3z^2$ in $(x+y+z)^{10}$
(ii) x^3yz^4t in $(x+y+z+t)^9$.

6. Let p be a prime. Show that the multinomial number $\binom{p}{n_1,\, n_2,\, \ldots,\, n_k}$ is divisible by p unless one of the n_i is equal to p.

Section 12.4 (Partitions of a positive integer)

1. Write down all the partitions of 7.

2. Let $p(n,\, k)$ denote the number of partitions of n into (exactly) k parts. Prove that
$$p(n,\, k) = \sum_{j=1}^{k} p(n-k,\, j).$$

3. With the help of Ex. 2, or otherwise, write out the first seven rows of the "partition triangle".

Review Section 12.7

1. How many 14-letter words can be made from the letters of the word CLASSIFICATION ?

2. Calculate the coefficient of

(i) $x^3y^2z^4$ in $(x+y+z)^9$
(ii) xy^3zt^2u in $(x+y+z+t+u)^8$.

3. Calculate $p(8)$, the total number of partitions of 8, and verify that the number which have distinct parts is equal to the number whose parts are all odd. Can you explain this equality (which holds for any n, not just $n=8$) ?

5. Show that $S(n,\,3)=\frac{1}{2}(3^{n-1}+1)-2^{n-1}$.

6. Let $\sim$ denote the relation on $\mathbb{Z}$ defined by

$$a\sim b \quad\Leftrightarrow\quad a-b \text{ is divisible by } 11.$$

Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. How many equivalence classes are there ?

8. Prove that

$$\sum_{k=1}^{m} S(m,\,k)\times\prod_{j=1}^{k}(n-j+1)=n^m.$$

10. Let q_n denote the total number of partitions of an n-set. Prove that

$$q_n=\sum_{k=1}^{n}\binom{n-1}{k-1}q_{n-k}=\sum_{k=0}^{n-1}\binom{n-1}{k}q_k.$$

11. Use the Inclusion-Exclusion principle to show that the number of surjections from an n-set to a k-set is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

12. Prove that

$$S(n,\,k)=\frac{1}{k!}\sum_{n_1+\cdots+n_k=n;\,n_i>0}\binom{n}{n_1,\,n_2,\,\ldots,\,n_k}.$$

13. Show that if the sum on the right-hand side in Ex. 12 (without the $1/k!$ factor) is taken over all *non-negative* integers satisfying $n_1+\cdots+n_k=n$, then the result is k^n.

14. In how many ways can mn distinguishable objects be distributed among m identical boxes so that there are n objects in each box ?

15. By using the multinomial numbers show that, for any positive integer n,

(i) 2^n divides $(2n)!$, and the quotient is even for $n \geq 2$;
(ii) $(n!)^{n+1}$ divides $(n^2!)!$.