ANSWERS: CHAPTER 12

Section 12.1 (Partitions of a set)

1. We employ the recursion for $S(n,\, k)$ derived in the lecture notes. The eighth row is: $1,\, 127,\, 966,\, 1701,\, 1050,\, 266,\, 28,\, 1$.

2. First take $k = 2$. A partition of an n-set X into two parts is just a pair $(A,\, B)$ such that $A \cup B = X$, $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$. In other words, A is any subset of X other than ϕ or X itself and $B = A^c$. There are $2^n - 2$ choices for A. But the pairs $(A,\, A^c)$ and $(A^c,\, A)$ represent the same partition, hence $S(n,\, 2) = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$, v.s.v.

Next consider $k = n - 1$. If we partition an n-set X into $n - 1$ parts, then exactly one of the parts will contain two elements, and all other parts will contain exactly one element. Hence the partition is determined completely by the choice of the two elements to place in the same part. Thus $S(n,\, n-1)$ equals the number of 2-element subsets of X, which is $\binom{n}{2}$.

3. We fix $x \in X$. If the part containing x contains $n - r$ elements then there are $\binom{n-1}{n-1-r} = \binom{n-1}{r}$ choices for it, since we have to choose which $n - 1 - r$ elements to put in the same part as x. Having done so, we have r elements of X left to place into $k - 1$ parts, which can be done in $S(r,\, k-1)$ ways. By MP + AP, it follows that $S(n,\, k) = \sum_{r=0}^{n-1} \binom{n-1}{r} S(r,\, k-1)$.

Section 12.2 (Classification and equivalence relations)

1. It's a standard exercise to show that, for any $n \in \mathbb{N}$, the relation "$x \sim x'$ if n divides $x - x'$" is an equivalence relation on $\mathbb{Z}$, and that there are n equivalence classes, the set of which is denoted $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$. You may have already seen this in some earlier course such as elementary nuber theory.

In the present exercise, the equivalence classes are $\{5\}$, $\{1,\, 6,\, 11\}$, $\{2,\, 7\}$, $\{9\}$.

2. **(i)** $4! = 24$. **(ii)** Easy. **(iii)** $24/4 = 6$. The easiest way to get representatives for different classes is to fix one number and permute the other three, in $3! = 6$ ways. Thus, for example, a set of representatives is given by $(1,\, 2,\, 3,\, 4)$, $(1,\, 3,\, 2,\, 4)$, $(2,\, 1,\, 3,\, 4)$, $(2,\, 3,\, 1,\, 4)$, $(3,\, 1,\, 2,\, 4)$, $(3,\, 2,\, 1,\, 4)$.

3. $n!$ seating plans and $n!/n = (n-1)!$ equivalence classes.

4. Not reflexive since 0 is not related to itself. It is symmetric since multiplication is commutative: $nn' = n'n$. It is also transitive: if $nn' > 0$ and $n'n'' > 0$ then $n' \neq 0$

and $nn'' = [(nn')(n'n'')]/(n')^2 > 0$, since $(n')^2 > 0$.

5. The argument assumes that for every a, there exists at least one b such that $a \sim b$. So it breaks down for any a which is not related to anything.

Section 12.3 (Distributions and multinomial numbers)

1. $\binom{11}{4,\,2,\,2,\,1} = \frac{11!}{4!\,2!\,2!} = 415800.$

2. $\binom{10}{4,\,3,\,2,\,1} = \frac{10!}{4!\,3!,\,2!} = 12600.$ $\binom{9}{5,\,2,\,2} = \frac{9!}{5!\,2!\,2!} = 756.$

3. After four moves there will be two crosses (X), two circles (O) and five blank spaces (B). Hence the number of possibilities equals the number of words that can be made from XXOOBBBBB, which is $\binom{9}{5,\,2,\,2} = 756.$

4. Same idea as in the proof of Pascal's identity (2.2). The LHS can be interpreted as the number of words that can be made from n letters, a of which are A, b of which are B and c of which are C. The three terms on the RHS can be interpreted as the number of words in which the first letter is A, B or C respectively.

The more general formula is: if $n_1 + \cdots + n_k = n$ then

$$\binom{n}{n_1,\, n_2,\, \ldots,\, n_k} = \sum_{i=1}^{k} \binom{n-1}{n_1,\, \ldots,\, n_i - 1,\, \ldots,\, n_k}.$$

5. (i) $\binom{10}{5,\,3,\,2} = \frac{10!}{5!\,3!\,2!} = 2520.$ **(ii)** $\binom{9}{3,\,1,\,4,\,1} = \frac{9!}{3!\,4!} = 2520$ (same answer !).

6. We can write it as $\frac{p!}{n_1!\,n_2!\ldots n_k!}$. The numerator contains p as a factor. Since $n_1 + \cdots + n_k = p$, none of the $n_i!$ contains p as a factor unless $k = 1$ and $n_1 = p$. In all other cases, the denominator is not divisible by p (since p is prime), hence the quotient remains divisible by p, v.s.v.

Section 12.4 (Partitions of a positive integer)

1.

$$7,\quad 6+1,\quad 5+2,\quad 5+1+1,\quad 4+3,\quad 4+2+1,$$
$$4+1+1+1,\quad 3+3+1,\quad 3+2+2,\quad 3+2+1+1,\quad 3+1+1+1+1,$$
$$2+2+2+1,\quad 2+2+1+1+1,\quad 2+1+1+1+1+1,\quad 1+1+1+1+1+1+1.$$

Hence $p(7) = 15$.

2. Given a partition of n into k parts, do the following: (i) remove all the parts which are 1 (ii) subtract 1 from the remaining parts. This will yield a partition of $n-k$ into j parts, where j is the number of parts in the original partition which are greater than

1. This procedure is clearly reversible: given a partition of $n-k$ into j parts, where $j \le k$, we can add one to each part and then append $k-j$ ones to get a partition of n into exactly k parts.

Hence, there is a $1-1$ correspondence between partitions of n into exactly k parts and partitions of $n-k$ into at most k parts, which is what the desired recursion says.

3.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 2 & & 1 & & 1 & & \\
 & 1 & & 3 & & 3 & & 2 & & 1 & & 1 & \\
1 & & 3 & & 4 & & 3 & & 2 & & 1 & & 1
\end{array}$$

Review Section 12.7

1. $\binom{14}{3,\,2,\,2,\,2,\,1,\,1,\,1,\,1,\,1} = \frac{14!}{3!\,(2!)^3} = 1,816,214,400.$

2. (i) $\binom{9}{3,\,2,\,4} = \frac{9!}{3!\,2!\,4!} = 2520.$ **(ii)** $\binom{8}{1,\,3,\,1,\,2,\,1} = \frac{8!}{3!\,2!} = 3360.$

3. The eighth row of the partition triangle can be computed as: $1,\ 4,\ 5,\ 6,\ 3,\ 2,\ 1,\ 1$. Hence $p(8) = 22$. There are six partitions with distinct parts, namely

$$8, \quad 7+1, \quad 6+2, \quad 5+3, \quad 5+2+1, \quad 4+3+1,$$

and also six partitions with odd parts, namely

$$7+1,\ \ 5+3,\ \ 5+1+1+1,\ \ 3+3+1+1,\ \ 3+1+1+1+1+1,\ \ 1+1+1+1+1+1+1+1.$$

It is a famous theorem of Euler that the number of partitions of n with distinct parts always equals the number of partitions of n into odd parts. The simplest (known) proof involves generating functions, though there are also known explicit descriptions of bijections between the two sets of partitions. For a discussion see, for example,

`https://shreevatsa.wordpress.com/2008/10/15/a-theorem-by-euler-on-partitions/`

5. From Ex. 12.1.2 we know that $S(n,\,2) = 2^{n-1} - 1$. We also have the recursion $S(n,\,k) = S(n-1,\,k-1) + k \cdot S(n-1,\,k)$ which, for $k = 3$, states that

$$S(n,\,3) = S(n-1,\,2) + 3 \cdot S(n-1,\,3) = 2^{n-2} - 1 + 2 \cdot S(n-1,\,3) \quad \forall\, n \ge 2. \quad (0.1)$$

We can thus prove the desired formula by induction on n. It is true for $n = 1$ since $0 = S(1,\,3) = \frac{1}{2}(3^0 - 1) - 2^0$. Suppose it is true for $n = m$ and consider $n = m + 1$.

Then (0.1) gives

$$S(m+1,\, 3) = 2^{m-1} - 1 + 3\left(\frac{1}{2}(3^{m-1}+1) - 2^{m-1}\right) =$$

$$\frac{1}{2}(3^m) + \left(\frac{3}{2} - 1\right) + (2^{m-1} - 3 \cdot 2^{m-1}) = \frac{1}{2}(3^m + 1) - 2^m, \quad \text{v.s.v.}$$

6. See Ex. 12.2.1.

8. Note that $\prod_{j=1}^{k}(n-j+1) = P(n,\, k)$, so the formula says that $\sum_{k=1}^{m} S(m,\, k)\, P(n,\, k) = n^m$. The RHS is the total number of functions $f : X \to Y$, where $|X| = m$ and $|Y| = n$. I claim that $S(m,\, k)\, P(n,\, k)$ equals the number of such functions for which the range has size k, i.e.: $|f(X)| = k$. For if the range has size k, say $f(x) = \{y_1,\, y_2,\, \ldots,\, y_k\}$, then we have divided X into k parts, namely $f^{-1}(y_1),\, f^{-1}(y_2),\, \ldots,\, f^{-1}(y_k)$ and mapped these parts to $y_1,\, y_2,\, \ldots,\, y_k$ respectively, i.e.: to an ordered choice of k elements from Y. The partition can be made in $S(m,\, k)$ ways and the ordered choice in $P(n,\, k)$ ways, so MP implies what we want.

10. Note that the two sums are the same thing, just reindexed $k \leftrightarrow n-k$. So it suffices to prove that $q_n = \sum_{k=0}^{n-1}\binom{n-1}{k} q_k$. By definition, $q_n = \sum_{k=0}^{n} S(n,\, k)$. Now apply Ex. 12.1.3 and interchange the order of summation:

$$q_n = \sum_{k=0}^{n} S(n,\, k) = \sum_{k=0}^{n}\sum_{r=0}^{n-1} S(r,\, k-1) =$$

$$= \sum_{r=0}^{n-1}\binom{n-1}{r}\sum_{k=0}^{n} S(r,\, k-1) = \sum_{r=0}^{n-1}\binom{n-1}{r}\sum_{l=0}^{r} S(r,\, l) = \sum_{k=0}^{n-1}\binom{n-1}{r} q_r, \quad \text{v.s.v.}$$

11. See Homework 1.

12. Fix a k-tuple $(n_1,\, n_2,\, \ldots,\, n_k)$ of positive integers whose sum is n. The multinomial coefficient is the number of words that can be made from n letters, n_i of which are the "letter" L_i, for $i = 1,\, 2,\, \ldots,\, k$. Given such a word, we can extract from it an ordered partition $(X_1,\, X_2,\, \ldots,\, X_k)$ of $\{1,\, 2,\, \ldots,\, n\}$ into k parts, by reading the word from left-to-right and letting X_i consist of the positions in which the letter L_i occurs. Clearly this idea provides a $1-1$ correspondence between "words of length n consisting of k distinct letters" and "ordered partitions of an n-set into k parts". However, the Stirling numbers count *unordered* partitions of an n-set, so we must divide by $k!$ to get $S(n,\, k)$, which is exactly what the desired formula says.

13. Using the same idea as in Ex. 12, if we sum over all non-negative k-tuples $(n_1,\, n_2,\, \ldots,\, n_k)$, then we are counting all possible words of length n in the letters $L_1,\, L_2,\, \ldots,\, L_k$, where now there is no requirement to use every letter at least once. There is a $1-1$ corrspondence between such words and functions $f : \{1,\, 2,\, \ldots,\, n\} \to \{L_1,\, L_2,\, \ldots,\, L_k\}$ - we interpret $f(j)$ as the j:th letter in the word. Hence the sum is just the total number of such functions, which by MP is k^n, v.s.v.

14. $\frac{1}{m!}\binom{mn}{n, n, \ldots, n} = \frac{(mn)!}{m!\,(n!)^m}$.

15. **(i)** From Ex. 14 with $m = 2$ it follows that $\frac{(2n)!}{n!\,(2!)^n}$ is always an integer. Hence $\frac{(2n)!}{2^n}$ is an integer divisible by $n!$, thus divisible by 2 for all $n \geq 2$.

(ii) The result is obviously true for $n = 1$, so suppose $n \geq 2$. Then, at the very least, $(n^2)! \geq n(n+1)$. Note that, if $a \leq b$ are positive integers, then $a!$ divides $b!$. Hence it suffices to show that $(n!)^{n+1}$ divides $[n(n+1)]!$. But this follows from the fact that $\frac{[n(n+1)]!}{(n!)^{n+1}} = \binom{n(n+1)}{n, n, \ldots, n}$, which must be an integer.