EXERCISES: CHAPTER 15

Section 15.1 (Graphs and their representation)

1. Three houses A, B, C each have to be connected to the gas, water and electricity supplies: G, W, E. Write down the adjacency list and matrix for the graph which represents this problem, and make a pictorial representation of it. Can you find a picture in which the lines representing the edges do not cross ?

2. The pathways in a formal garden are to be laid out in the form of a *wheel graph* W_n, whose vertex set is $V = \{0, 1, \ldots, n\}$ and whose edges are

$$\{0, 1\}, \quad \{0, 2\}, \ldots, \{0, n\},$$
$$\{1, 2\}, \quad \{2, 3\}, \ldots, \{n-1, n\}, \quad \{n, 1\}.$$

Draw the graph and describe a route around the pathways which starts and ends at vertex 0 and visits every vertex once only.

3. How many edges has the complete graph K_n ? For which values of n can you find a pictorial representation of K_n with the property that the lines representing the edges do not cross ?

4. A 3-*cycle* in a graph is a set of three mutually adjacent vertices. Construct a graph with five vertices and six edges which contains no 3-cycles.

Section 15.2 (Isomorphism of graphs)

1. Prove that the graphs in Figure 15.2.1 are not isomorphic.

2. Find an isomorphism between the graphs defined by the following adjacency lists.

a	b	c	d	e	f	g	h	i	j
b	a	b	c	d	a	b	c	d	e
e	c	d	e	a	h	i	j	f	g
f	g	h	i	j	i	j	f	g	h

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	0	1	0	2	6
5	0	1	2	3	4	4	3	5	7
7	6	8	7	6	8	9	9	9	8

3. Let $G = (V, E)$ be the graph defined as follows. The vertex set V is the set of all words of length 3 in the alphabet $\{0, 1\}$, and the edge set E contains those pairs of words which differ in exactly one position. Show that G is isomorphic to the graph formed by the corners and edges of an ordinary cube.

Section 15.3 (Degree)

1. Is it possible that the following lists are the degrees of all the vertices of a graph ? If so, give a pictorial representation of such a graph.

(i) $2, 2, 2, 3$
(ii) $1, 2, 2, 3, 4$
(iii) $2, 2, 4, 4, 4$
(iv) $1, 2, 3, 4$.

2. If $G = (V, E)$ is a graph, the *complement* $\overline{G}$ of G is the graph whose vertex set is V and whose edges join those pairs of vertices which are not joined in G. If G has n vertices and their degrees are $d_1, d_2, \ldots, d_n$, what are the degrees of the vertices of $\overline{G}$?

3. Find as many different (non-isomorphic) regular graphs with degree 4 and seven vertices as you can.
(HINT: Consider the complement of such a graph).

5. Show that if G is a graph with at least two vertices then G has two vertices with the same degree.

Section 15.4 (Paths and cycles)

1. Find the number of components of the graph whose adjacency list is

a	b	c	d	e	f	g	h	i	j
f	c	b	h	c	a	b	d	a	a
i	g	e		g	i	c		f	f
j		g			j	e			

2. Professor McBrain and his wife April give a party at which there are four other married couples. Some pairs of people shake hands when they meet, but naturally no couple shake hands with each other. At the end of the party the Professor asks everyone how many people they have shaken hands with, and he receives nine different answers.

(i) How many people shook hands with April ?

(ii) Let $G = (V,\, E)$ be the graph whose vertices are the ten guests and where an edge is placed between each pair that shook hands. How many components has G ?

3. Find a Hamiltonian cycle in the graph formed by the vertices and edges of an ordinary cube.

4. Next year Dr. X and Dr. Y intend to visit the island of Hisingen, where the interesting places and the roads joining them are represented by the graph whose adjacency list is

0	1	2	3	4	5	6	7	8
1	0	1	0	3	0	1	0	1
3	2	3	2	5	4	5	2	3
5	6	7	4		6	7	6	5
7	8		8		8		8	7

Dr. X is a tourist by nature, and wishes to visit each place once and return to his starting point. Dr. Y is an explorer, and wishes to traverse every road just once, in either direction; she is prepared to start and finish in different places. Is it possible to find routes to suit (either of) them ?

5. A mouse intends to eat a $3 \times 3 \times 3$ cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a $1 \times 1 \times 1$ cube before going on to an adjacent one. Can the mouse end in the centre ?

Section 15.5 (Trees)

1. There are six different (that is, mutually non-isomorphic) trees with six vertices: draw them !

2. Let $T = (V,\, E)$ be a tree with $|V| \geq 2$. Show that T has at least two vertices of degree 1, i.e.: two so-called *leaves*.

3. Show that, if $G = (V,\, E)$ is a graph then property (1) implies both (2) and (3), where

(1): for each pair $x,\, y$ of vertices there is a unique path in G from x to y
(2): G is connected
(3): there are no cycles in G.

4. A *forest* is a graph each of whose connected components is a tree. Prove that if $F = (V,\, E)$ is a forest with c components, then $|E| = |V| - c$.

Section 15.6 (Coloring the vertices of a graph)

1. Find the chromatic numbers of the following graphs:

(i) the complete graph K_n
(ii) a cycle graph C_{2r} with an even number of vertices
(iii) a cycle graph C_{2r+1} with an odd number of vertices.

2. Determine the chromatic numbers of the graphs depicted in Figure 15.6.2.

3. Describe all graphs G for which $\chi(G) = 1$.

Section 15.7 (The greedy algorithm for vertex coloring)

1. Find orderings of the vertices of the cube graph (see the solution to Exercise 15.2.3) for which the greedy coloring algorithm requires 2, 3 and 4 colors respectively.

2. Show that for any graph G there is an ordering of the vertices for which the greedy algorithm requires just $\chi(G)$ colors.

3. Let $e_i(G)$ denote the number of vertices of a graph G whose degree is strictly greater than i. Use the greedy algorithm to show that if $e_i(G) \leq i + 1$ for some i, then $\chi(G) \leq i + 1$.

4. The graph M_r $(r \geq 2)$ is obtained from the cycle graph C_{2r} by adding extra edges joining each pair of opposite vertices. Show that

(i) M_r is bipartite when r is odd
(ii) $\chi(M_r) = 3$ when r is even and $r \neq 2$
(iii) $\chi(M_2) = 4$.

Review Section 15.8

1. For which values of n does the complete graph K_n have an Eulerian walk ?

2. Use induction to show that, if $G = (V, E)$ is a graph with $|V| = 2m$, and G has no 3-cycles, then $|E| \leq m^2$.

3. Let $X = \{1, 2, 3, 4, 5\}$ and let $V = \binom{X}{2}$. Let E denote the set of pairs of members of V which are disjoint, as subsets of X. Show that the graph $G = (V, E)$ is isomorphic to the graph of Figure 15.8.3. Also show that both are isomorphic to the graphs of Exercise 15.2.2, and in turn to Petersen's graph.

4. Let G be a bipartite graph with an odd number of vertices. Show that G cannot have a Hamiltonian cycle.

5. The *k-cube* Q_k is the graph whose vertices are the words of length k in the alphabet $\{0,\,1\}$ and whose edges join words that differ in exactly one position. Show that

(i) Q_k is regular of degree k
(ii) Q_k is bipartite.

6. Prove that Q_k has a Hamiltonian cycle, for all $k \geq 2$.

7. Show that Petersen's graph does not have a Hamiltonian cycle.

8. In a game of dominoes (see Exercise 10.7.2) the rules require that the dominoes be placed in a line so that adjacent dominoes have matching numbers: $[x|y]$ is next to $[y|z]$ and so on. By regarding the dominoes for which $x \neq y$ as the edges of the complete graph K_7, show that it is possible to have a game in which all the dominoes are used.

11. If G is a regular graph with degree k and n vertices, show that $\chi(G) \geq \frac{n}{n-k}$.

12. Construct five mutually non-isomorphic connected regular graphs with degree 3 and eight vertices.

13. Show that the complete graph K_{2n+1} is the union of n Hamiltonian cycles, no two of which have a common edge.

14. Is it possible for the knight to visit all the squares of a chessboard exactly once and return to its starting square ?

15. The *odd graph O_k* is defined as follows, when $k \geq 2$: the vertices are the $(k-1)$-subsets of a $(2k-1)$-set, and the edges join disjoint subsets. Show that

(i) O_3 is isomorphic to Petersen's graph
(ii) $\chi(O_k) = 3$ for all $k \geq 2$.

16. Show that if G is a graph with n vertices, m edges and c connected components, then

$$n - c \leq m \leq \frac{1}{2}(n-c)(n-c+1).$$

Construct examples showing that both bounds can be attained, for all values of n and c such that $n \geq c$.

17. A sequence $d_1,\,d_2,\,\ldots,\,d_n$ is said to be *graphic* if there is a graph whose n vertices can be labelled $v_1,\,v_2,\,\ldots,\,v_n$ in such a way that deg$(v_i) = d_i$, $i = 1,\,2,\,\ldots,\,n$. Show that if $d_1,\, d_2,\, \ldots,\, d_n$ is a graphic sequence and $d_1 \geq d_2 \geq \cdots \geq d_n$ then

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k,\, d_i\}, \quad \forall\, k = 1,\, \ldots,\, n.$$

18. The *girth* of a graph G is the least value of g for which G contains a g-cycle. Show that a regular graph with degree k and girth $2m+1$ must have at least

$$1 + k + k(k-1) + \cdots + k(k-1)^{m-1}$$

vertices, and that a regular graph with degree k and girth $2m$ must have at least

$$2[1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{m-1}]$$

vertices.

19. Construct a table of the lower bounds obtained in the previous exercise when $k = 3$ and the girth is $3,\, 4,\, 5,\, 6,\, 7$. Show that there is a graph which attains the lower bound in the first four cases, but not in the fifth.

21. Let $G = (V,\, E)$ be a graph with at least three vertices such that $\deg(v) \geq \frac{1}{2}|V|$ for all $v \in V$. Prove that G contains a Hamiltonian cycle.