ANSWERS: CHAPTER 15

Section 15.1 (Graphs and their representations)

1. For the picture, see Figure 15.1.1. It is not possible to draw the graph in such a way that edges don't cross, in other words $K_{3,3}$ is not planar. See the discussion in the lecture notes on Kuratowski's theorem. The adjaceny list and matrix are given below. In the latter, the rows/columns are indexed left-to-right/top-to-bottom as A, B, C, G, W, E.

A	B	C	G	W	E
G	G	G	A	A	A
W	W	W	B	B	C
E	E	E	C	C	C

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

2. For the drawing, see Figure 15.1.2. A suitable path is $0 \to 1 \to 2 \to \cdots \to n \to 0$.

3. It has $\binom{n}{2}$ edges and is planar iff $n \leq 4$, see the lecture notes.

4. For example, take $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}$. See Figure 15.1.4.

Section 15.2 (Isomorphism of graphs)

1. For example, the second graph has no 3-cycles.

2. An example of an isomorphism is the mapping

$$a \mapsto 0, \quad b \mapsto 7, \quad c \mapsto 9, \quad d \mapsto 6, \quad e \mapsto 1, \quad f \mapsto 5, \quad g \mapsto 3, \quad h \mapsto 8, \quad i \mapsto 4, \quad j \mapsto 2.$$

To see this, first draw the two graphs as in Figure 15.2.2, and then use Exercise 15.8.3.

3. See Figure 15.2.3.

Section 15.3 (Degree)

1. (i) Not possible, since the sum of the degrees must be even.
(ii) Possible. See Figure 15.3.1(ii).
(iii) Not possible. Such a graph would have 5 vertices and three of them would be connected to every other vertex. This would imply that every vertex had degree at least 3.
(iv) Not possible. If G has n vertices, then no degree can exceed $n-1$.

2. $n-1-d_1,\ n-1-d_2,\ \dots,\ n-1-d_n$.

3. $G \cong H$ iff $\overline{G} \cong \overline{H}$. If G has seven vertices, each of degree 4, then $\overline{G}$ has seven vertices, each of degree 2. Hence, every component of $\overline{G}$ is a cycle, which leaves two possibilities: (i) $\overline{G}$ is a 7-cycle (ii) $\overline{G}$ has two components, a 3-cycle and a 4-cycle.

So there are two non-isomorphic graphs on seven vertices that are regular of degree 4. They are illustrated in Figure 15.3.3.

5. Each degree is one of the integers $0,\ 1,\ \dots,\ n-1$ and there are n vertices.

Case 1: G has no isolated vertex, i.e.: no vertex of degree 0. Then there are only $n-1$ possibilities for the degree of any vertex, and there are n vertices, so by the Pigeonhole Principle there must be two vertices with the same degree.

Case 2: G has at least two isolated vertices. Then these all have the same degree, namely 0.

Case 3: G has exactly one isolated vertex. This leaves $n-1$ vertices and each has degree at least 1. But each has degree at most $n-2$, since none of them connects to the isolated vertex. Hence there are only $n-2$ possibilities for the degrees of the non-isolated vertices and $n-1$ such vertices, so PP implies again that two must have the same degree.

Section 15.4 (Paths and cycles)

1. There are three components, $\{a,\ f,\ i,\ j\}$, $\{b,\ c,\ e,\ g\}$ and $\{d,\ h\}$. See Figure 15.4.1.

2. (i) There are a total of five married couples, hence 10 guests, at the party. The nine answers gotten by the Professor must have been $0,\ 1,\ \dots,\ 8$. We identify these numbers with the corresponding people. Nr. 8 must have shaken the hand of everyone except his/her own spouse (and him/her self). In particular, everyone other than his/her spouse must have shaken at least one hand. Hence Nr. 8 must be married to Nr. 0. Next consider Nr. 7. There are two people whose hands (s)he didn't shake, and these two must be Nr. 0 and his/her own spouse. In particular, everyone other than Nr. 0 and Nr. 7's spouse must have shaken the hand of both Nrs. 7 and 8, which implies that Nr. 7 must be married to Nr. 1.

One can continue in the same way (details left to reader to write out) to conclude that Nr. 6 is married to Nr. 2 and Nr. 5 to Nr. 3, which leaves Nr. 4 as the spouse of the Professor. Hence, April shook 4 hands.

(ii) The graph must have two components, since Nr. 0 is an isolated vertex whereas Nr. 8 is joined to everyone else.

3. For example,

$$(0,\,0,\,0) \to (1,\,0,\,0) \to (1,\,1,\,0) \to (0,\,1,\,0) \to (0,\,1,\,1) \to \\ \to (1,\,1,\,1) \to (1,\,0,\,1) \to (0,\,0,\,1) \to (0,\,0,\,0).$$

4. There is no solution for Dr. X since the graph has no Hamiltonian cycle. It *does* have a Hamiltonian path, for example $8 \to 7 \to 6 \to 5 \to 0 \to 1 \to 2 \to 3 \to 4$, but then there's no edge from 4 back to 8. Basically, vertex 4 is the bottleneck.

There is a solution for Dr. Y. We can see immediately from the adjacency list that there are exactly two vertices of odd degree, namely 2 and 6, so there exists an Eulerian walk between these two.

5. It is impossible. The $3^3 = 27$ small cubes making up the big cube can be represented by a graph $G = (V,\,E)$, where V consists of all points $(a_1,\,a_2,\,a_3) \in \mathbb{Z}^3$ such that $a_i \in \{0,\,1,\,2\}$, and where an edge is placed between two points if and only if they are adjacent in the lattice, i.e.: iff they differ by exactly one in exactly one coordinate. This graph is bipartite since $V = V_e \sqcup V_o$, where V_e (resp. V_o) consists of the points for which $a_1 + a_2 + a_3$ is even (resp. odd), and every edge goes between a vertex in V_e and one in V_o. There are 14 vertices in V_e and 13 in V_o. This implies that any Hamiltonian path in G must start and end in V_e, since the path goes back and forth between V_e and V_o. But the middle of the cube is represented by $(1,\,1,\,1)$, a point in V_o. Hence the mouse cannot end there, no matter where he starts from.

Section 15.5 (Trees)

1. See Figure 15.5.1. Note that a methodical way to approach this problem is to start from a vertex of maximum degree, which can be 5, 4, 3 or 2.

2. Let $|V| = n$. Then $|E| = n - 1$. Let the degrees of the vertices be $d_1,\,d_2,\,\ldots,\,d_n$. We have $\sum_{i=1}^{n} d_i = 2|E| = 2n - 2$. But each $d_i \geq 1$, since T is connected. Hence, at least two of the d_i must equal one, as otherwise the degree sum would be at least $2n - 1$.

3. (1) $\Rightarrow$ (2): Well, (2) says that, for any pair $x,\,y$ of vertices, there is *some* path between them in G. (1) says this and more, namely that for each pair $x,\,y$, such a path is unique.

(1) $\Rightarrow$ (3): Suppose (3) did not hold. Let C be any cycle in G and imagine the vertices along it being drawn in a circle. Let $x,\,y$ be any pair of vertices along the cycle. Then there are at least two distinct paths between x and y, namely we can go either clockwise or anti-clockwise along the cycle C. This contradicts (1).

4. Let the components be $F_i = (V_i, E_i)$, $i = 1, \dots, c$. Each F_i is a tree, hence $|E_i| = |V_i| - 1$. But $|E| = \sum_i |E_i| = \sum_i (|V_i| - 1) = (\sum_i |V_i|) - c = |V| - c$, v.s.v.

Section 15.6 (Coloring the vertices of a graph)

1. $\chi(K_n) = n$, $\chi(C_{2r}) = 2$ and $\chi(C_{2r+1}) = 3$.

2. (i) $\chi(G) = 3$. A 3-coloring is illustrated in Figure 15.6.2(i). We must prove it is impossible to color with only two colors. If one tries to do so, then because of the outer 8-cycle, the colors would need to alternate along this cycle. But then opposite vertices would get the same color - contradiction, since opposite vertices are joined in G.

(ii) $\chi(G) = 4$. A 4-coloring is illustrated in Figure 15.6.2(ii). We must prove it is impossible to color with 3 colors. Suppose one tried to so, with colors a, b, c. The outer 5-cycle must use all 3 colors and, up to rotations and permutations of the colors, the only possibility is, reading clockwise from the top vertex, a, b, a, b, c. One checks that there will then be three arms of the inner 5-star whose ends have respectively the pairs of colors $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. The inner vertices along these arms must therefore get the colors c, b, a respectively. But then the innermost vertex must get a fourth color.

(iii) $\chi(G) = 4$. A 4-coloring is illustrated in Figure 15.6.2(iii). We must prove it is impossible to color with 3 colors. Say we try with colors a, b, c. The top two vertices along with that on the right form a K_3, hence these must get three different colors. WLOG, assign these three vertices the colors a, b, c, reading clockwise. Continuing clockwise, the two bottom vertices must get colors a, b, since these are both joined to the vertex on the right, as well as vertically upwards. But then the leftmost vertex is already joined to three vertices who've received colors a, b, c, so it must get a fourth color.

3. $\chi(G) = 1$ iff G is a collection of isolated vertices, i.e.: iff G has no edges at all.

Section 15.7 (The greedy algorithm for vertex coloring)

1. Consider the following three orderings:

$$\mathcal{O}_1 = \{(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1), (0, 1, 1), (0, 0, 1), (1, 0, 1)\},$$
$$\mathcal{O}_2 = \{(1, 0, 0), (0, 1, 1), (1, 1, 1), (1, 0, 1), (0, 0, 1), (0, 0, 0), (1, 1, 0), (0, 1, 0)\},$$
$$\mathcal{O}_3 = \{(1, 0, 0), (0, 1, 1), (1, 1, 1), (1, 0, 1), (0, 0, 0), (0, 0, 1), (1, 1, 0), (0, 1, 0)\}.$$

If we color greedily with colors $a,\ b,\ c,\ d$ then, in these three orderings, the colors assigned will be

$$\text{For } \mathcal{O}_1\text{: } a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,$$
$$\text{For } \mathcal{O}_2\text{: } a,\ a,\ b,\ c,\ b,\ c,\ c,\ b,$$
$$\text{For } \mathcal{O}_3\text{: } a,\ a,\ b,\ c,\ b,\ d,\ c,\ d.$$

2. There exists, by defintion, *some* coloring of G using exactly $\chi(G)$ colors. Consider any such coloring $\mathcal{C}$, using colors $i = 1,\ 2,\ \ldots,\ \chi(G)$, say. Now order the vertices of G in such a way that the vertices assigned color j all come before those assigned color k, whenever $j < k$. Vertices assigned the same color can be ordered amongst themselves arbitrarily. Then the greedy algorithm will produce exactly the coloring $\mathcal{C}$.

3. Fix an i such that $e_i(G) \leq i + 1$. Consider any ordering of $V(G)$ such that all the vertices of degree $i + 1$ or more come first. Now color according to the greedy algorithm. A priori, at most $i + 1$ colors are used among the first $i + 1$ vertices in the ordering, i.e.: among the vertices of degree at least $i + 1$. Every subsequent vertex has degree at most i, which implies that at most i colors will have been used up by the vertices to which it's joined. This means that the greedy algorithm will always choose one of the first $i + 1$ available colors.

4. **(i)** If we denote the vertices, in clockwise order along the cycle, as $1,\ 2,\ \ldots,\ 2r$ then, when r is odd, every edge in M_r goes between a pair of vertices of opposite parity, so M_r is bipartite.

(ii) If r is even and $r \geq 2$, we can 3-color M_r by alternating with colors a and b on vertices $1,\ 2,\ \ldots,\ r$, switching to c for vertex $r + 1$, then back to alternating $a,\ b$ as far as vertex $2r - 1$, and finally using c again on vertex $2r$.

On the other hand, if we try to 2-color, then we *must* alternate with a and b on vertices $1,\ 2,\ \ldots,\ r$ and then we'll be forced to use a third color on vertex $r + 1$.

(iii) If $r = 2$ then $M_r = K_4$, and $\chi(K_4) = 4$.

Review Section 15.8

1. Every vertex in K_n has degree $n - 1$, so there exists an Eulerian cycle if and only if n is odd. If $n = 2$, there exists an Eulerian walk which is not a cycle.

2. If $m = 1$ and $|V| = 2$, then obviously $|E| \leq 1 = m^2$. Suppose the theorem is true for graphs on $2m$ vertices and let G be a graph on $2m + 2$ vertices. If G has no edges at all, then obviously $|E| \leq (m + 1)^2$. So pick any edge $\{v_1,\ v_2\}$. Let G' be the subgraph spanned by the remaining $2m$ vertices. If G has no 3-cycles then neither has G' so, by the induction hypothesis, G' has at most m^2 edges. Every other edge in G is either the edge $\{v_1,\ v_2\}$, or goes between one of these two vertices and a vertex in G'. But v_1 and v_2 can have no common neighbor in G', as otherwise we'd get a 3-cycle. Hence, the total number of edges in G is at most $1+|V(G')|+m^2 = 1+2m+m^2 = (m+1)^2$, v.s.v.

3. See Figure 15.8.3.

4. If G is bipartite, any path must cross back and forth between the two parts. If G has an odd number of vertices, then a Hamiltonian cycle contains an odd number of edges. But, in a bipartite graph, after an odd number of crossings we will be on the opposite side from where we started, so we can't have completed a cycle.

5. **(i)** Each vertex has k coordinates which can be switched to get an adjacent vertex. **(ii)** Every edge is between a vertex in V_e and one in V_o, where V_e (resp. V_o) is the set of vertices with an even (resp. odd) number of ones.

6. We proceed by induction on k. If $k = 2$, then $Q_k \cong C_4$, so is *is* a cycle. Suppose Q_k has a Hamiltonian cycle $\mathcal{C}_k$, WLOG starting and ending at $\mathbf{0}_k = (0, 0, \ldots, 0)$. Denote by $\mathcal{P}_k$ the corresponding Hamiltonian path, i.e.: leave out the last edge in $\mathcal{C}_k$ which returns to $\mathbf{0}_k$. In Q_{k+1}, we get a Hamiltonian cycle starting and ending at $\mathbf{0}_{k+1}$ as follows:

Step 1: Fix the first coordinate to be 0 and adjust the remaining k coordinates as if one were following the Hamiltonian path $\mathcal{P}_k$. Continue to the end of the path.

Step 2: Now switch the first coordinate to 1.

Step 3: Now adjust the remaining coordinates so that you follow the path $\mathcal{P}_k$ back to $\mathbf{0}_k$.

Step 4: Switch the first coordinate back to 0.

It is clear, I hope, that this describes a Hamiltonian cycle in Q_{k+1}.

7. This can be verified by exhaustive search. Note that the graph does possess simple cycles of length $5,\ 6,\ 7$ and 9.

8. This is just another way of saying that K_7 possesses an Eulerian walk, in fact it possesses an Eulerian cycle since every vertex has even degree, namely 6.

11. Suppose $\chi(G) < \frac{n}{n-k}$. Then there would be a vertex coloring of G using strictly less than $\frac{n}{n-k}$ colors. In such a coloring, at least one color would have to be used on strictly more than $n-k$ vertices, hence on at least $n-k+1$ vertices. These must form an independent set, call it S. Every neighbor of every vertex in S must be in the complement $V(G)\backslash S$. But there are only $k-1$ other vertices in G, hence every vertex in S has degree at most $k-1$. This contradicts the assumed k-regularity of G.

12. See Figure 15.8.12, which includes an explanation of why the five graphs are pairwise non-isomorphic.

13.

14. Yes, though I don't know of any simple, elegant way to prove it (i.e.: a way which avoids some form of brute-force search). If you're interested, see https://en.wikipedia.org/wiki/Knight's_tour

15. (i) See Exercise 15.8.3.
(ii) We first prove that $\chi(O_k) > 2$ by exhibiting an odd cycle in the graph. Let the underlying $(2k-1)$-set be $\{1, 2, \ldots, 2k-1\}$ and consider the following $2k$ vertices in O_k:

$$\begin{aligned} v_1 = \{1, 2, \ldots, k-1\}, \quad v_2 = \{k+1, k+2, \ldots, 2k-1\}, \\ v_3 = \{2, 3, \ldots, k\}, \quad v_4 = \{k+2, k+3, \ldots, 2k-1, 1\}, \\ v_5 = \{3, 4, \ldots, k+1\}, \quad v_6 = \{k+3, k+4, \ldots, 2k-1, 1, 2\}, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ v_{2k-1} = \{k, k+1, \ldots, 2k-2\}, \quad v_{2k} = v_1 = \{1, 2, \ldots, 2k-1\}. \end{aligned}$$

It's easy to see that each consecutive pair of $(k-1)$-sets is disjoint, hence $v_1 \to v_2 \to \cdots \to v_{2k-1} \to v_1$ is a $(2k-1)$-cycle in O_k.

To prove that $\chi(O_k) = 3$ it thus suffices to prove that

$$V(O_k) = V_1 \sqcup V_2 \sqcup V_3,$$

where each V_i is an independent set, i.e.: an intersecting collection of $(k-1)$-subsets of $\{1, 2, \ldots, 2k-1\}$. We let

$$\begin{aligned} V_1 &= \{X \subset \{1, 2, \ldots, 2k-1\} : |X| = k-1 \text{ and } 1 \in X\}, \\ V_2 &= \{X \subset \{1, 2, \ldots, 2k-1\} : |X| = k-1 \text{ and } 2 \in X\}, \\ V_3 &= \{X \subset \{1, 2, \ldots, 2k-1\} : |X| = k-1 \text{ and } X \subseteq \{3, 4, \ldots, 2k-1\}\}. \end{aligned}$$

It is obvious that V_1 and V_2 are intersecting families of sets. The same is true of V_3 since, if X, X' belong to V_3 then both are $(k-1)$-element subsets of a $(2k-3)$-set so $|X \cap X'| = |X| + |X'| - |X \cup X'| \geq (k-1) + (k-1) - (2k-3) \geq 1$.

16. The lower bound is achieved by a forest, see Exercise 15.5.4. For the upper bound, let $n_1, n_2, \ldots, n_c$ be the numbers of vertices in the various components. Then

$$n_i \geq 1 \,\forall\, i, \quad \sum_{i=1}^{c} n_i = n, \quad m \leq \sum_{i=1}^{c} \binom{n_i}{2} = \frac{1}{2}\sum_{i=1}^{c}(n_i^2 - n_i) = \frac{1}{2}\left(\sum_{i=1}^{c} n_i^2 - n\right).$$

Hence m is maximised by maximising $\sum_{i=1}^{c} n_i^2$, subject to the first two constaints. It is a simple calculus exercise (e.g.: using Lagrange multipliers) that the maximum is attained when $n_1 = n - c + 1$, $n_i = 1 \,\forall\, i > 1$. Hence,

$$m \leq \frac{1}{2}\left((n-c+1)^2 + (c-1)\cdot 1^2 - n\right) = \cdots = \frac{1}{2}(n-c)(n-c+1).$$

The maximum is attained when G is the union of a K_{n-c+1} and $c-1$ isolated vertices.

17. The sum on the left counts all ordered pairs (x, y) such that $\{x, y\} \in E(G)$ and $x \in \{v_1, v_2, \ldots, v_k\}$. Any pair for which also $y \in \{v_1, v_2, \ldots, v_k\}$ will be counted twice and there are at most $\binom{k}{2} = \frac{k(k-1)}{2}$ such pairs. This accounts for the first term on the right. Any pair for which y is not among the first k vertices will be counted once. The number of such pairs is at most $\sum_{i=k+1}^{n} \min\{k, d_i\}$ since each $y = v_j$ has no more than d_j neighbors in total, and a priori no more than k of them can be among $v_1, v_2, \ldots, v_k$. This accounts for the second term on the right.

18. **(i)** First suppose $g = 2m + 1$. Let v_0 be any vertex and set $N_0 := \{v_0\}$. Let N_1 be the set of its neighbors. Since G is k-regular, $|N_1| = v$. Let N_2 be the set of all neighbors of all vertices in N_1, other than v_0. Note that if any two vertices in N_1 were neighbors, then G would have a 3-cycle, formed by these two and v_0. Similarly, if any two vertices in N_1 had a common neighbor in N_2, then G would have a 4-cycle, formed by the two vertices in N_1, their common neighbor in N_2 and v_0. Hence, if $g \geq 5$, then $|N_2| = (k-1) \cdot |N_1|$.

Continue in the same manner: for $i = 1, \ldots, m$, define N_{i+1} to be the set of all neighbors of vertices in N_i other than those already in N_{i-1}. Then, since G has no cycles of length $2m$ or less, one has $|N_{i+1}| = (k-1) \cdot |N_i|$ for every $i = 1, \ldots, m$ and hence $|N_r| = (k-1)^{r-1}|N_1| = k(k-1)^{r-1}$, for $r = 1, \ldots, m$. The sets N_i are pairwise disjoint by definition, thus

$$|V(G)| \geq \sum_{r=0}^{m} |N_r| = 1 + k + k(k-1) + \cdots + k(k-1)^{m-1}, \quad \text{v.s.v.}$$

(ii) Suppose $g = 2m$. Let $\{v_0, w_0\}$ be any edge in G. Let $\mathcal{N}_{m-1}(v_0)$ denote the set of all vertices, other than w_0, which are at distance at most $m-1$ from v_0. Similarly, define $\mathcal{N}_{m-1}(w_0)$ to be the $(m-1)$-neighborhood of w_0, other than v_0. If these two sets had a vertex in common, say x, then G would contain a cycle of length at most $2m-1$, namely take a shortest path from v_0 to x, then a shortest path back to w_0 and finally the edge back to v_0.

Hence, the sets $\mathcal{N}_{m-1}(v_0)$ and $\mathcal{N}_{m-1}(w_0)$ must be disjoint, so in order to prove the result we're after it suffices to show that each of them contains at least $1 + (k-1) + \cdots + (k-1)^{m-1}$ vertices. But this is done in a similar manner to **(i)**. Namely, by an argument similar to that given in **(i)** one easily shows that, in $\mathcal{N}_{m-1}(v_0)$ say, there are at least $(k-1)^r$ vertices at distance exactly r from v_0, for $r = 0, 1, \ldots, m-1$. The reason for $(k-1)^r$ instead of $k(k-1)^{r-1}$ is because this time we've removed w_0 from the 1-neighborhood, so the analog of the set N_1 contains only $k-1$ vertices this time.

19. Following the argument in Exercise 18 leads us to what the extremal graphs must look like. See Figures 15.8.19 for examples when $g = 3, 4, 5, 6$ and an explanation of why it's not possible to make things work when $g = 7$.

21. See Dirac's theorem in the lecture notes.