EXERCISES: CHAPTER 17

Section 17.1 (Relations and bipartite graphs)

1. Let $X = \{2, 3, 5, 7, 11\}$, $Y = \{99, 100, 101, 102, 103\}$, and define x and y to be related whenever x is a divisor of y. Draw the bipartite graph representing this relation and verify that

$$\sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y) = |E|.$$

2. Let $K_{r,s} = (X, Y, E)$, the complete bipartite graph with $|X| = r$ and $|Y| = s$.
(i) What is the degree of each vertex in X ?
(ii) What is the degree of each vertex in Y ?
(iii) How many edges are there in $K_{r,s}$?
(iv) Describe in ordinary language a relation which $K_{r,s}$ could represent.
(v) Show that, for any $s \geq 1$, $K_{1,s}$ is a tree.
(vi) Show that $K_{r,s}$ is not a tree whenever $r \geq s \geq 2$.

3. Is the graph illustrated in Figure 17.1.3 bipartite ?

Section 17.2 (Edge colorings of graphs)

1. What is the least number of colors required for an edge-coloring of
(i) the complete graph K_4 ?
(ii) the complete graph K_5 ?
(iii) the cube graph Q_3 ?

2. Prove that there is no edge 3-coloring of Petersen's graph.

3. Prove that $K_{n,n}$ has an edge n-coloring, for every $n \geq 1$.

4. Show that the graph in Figure 17.2.4 is bipartite and construct an edge 3-coloring of it.

5. Show that Q_3 can be 3-edge colored. What about Q_k for general k ?

Section 17.4 (Matchings)

1. Use Hall's condition to show that the graph in Figure 17.4.1 has no complete matching.

2. Let M be the matching denoted by the wavy lines in Figure 17.4.1.
(i) Find an alternating path for M beginning at x_2.
(ii) Use it to construct a matching M' with $|M'| = 4$.
(iii) Is M' a maximum matching ?

3. Suppose that each member of a set of people has a list of k books which he or she wishes to borrow from a library. Suppose also that each book appears on exactly k lists. Show that it is possible for everyone to borrow one of the books on his or her list at the same time.

Section 17.5 (Maximum matchings)

1. Let $G = (X,\, Y,\, E)$ be the bipartite graph with $X = \{a,\, b,\, c,\, d,\, e\}$, $Y = \{v,\, w,\, x,\, y,\, z\}$ and $E = \{av,\, ax,\, bv,\, bz,\, cw,\, cy,\, cz,\, dy,\, dz,\, ez\}$. Use the augmenting path algorithm to find a complete matching in G, starting from the matching $M = \{av,\, bz,\, cy\}$.

2. Let $G = (X,\, Y,\, E)$ be the graph depicted in Figure 17.5.2. For which 3-subsets of X is it possible to find a maximum matching in G such that the three given vertices are matched ?

3. Suppose $G = (X,\, Y,\, E)$ is a bipartite graph with $|X| = |Y| = n$. Show that if δ is the minimum degree of a vertex then

$$|A| - |N(A)| \le n - \delta \quad \forall\, A \subseteq X.$$

Deduce that if $|E| > (m-1)n$ then G has a matching of size m.

Section 17.6 (Transversals for families of finite sets)

1. Let $\mathcal{S}$ be the family of sets

$$\{a,\, b,\, l,\, e\},\;\; \{l,\, e,\, s,\, t\},\;\; \{s,\, t,\, a,\, b\},\;\; \{s,\, a,\, l,\, e\},\;\; \{t,\, a,\, l,\, e\},\;\; \{s,\, a,\, l,\, t\}.$$

Find a transversal for $\mathcal{S}$.

2. Show that there is no transversal for the family $\mathcal{S}$ in Exercise 1 in which the first three sets are represented by $e,\, l,\, s$ respectively.

3. Prove that the family of sets

$$\{a,\, m\},\;\; \{a,\, r,\, e\},\;\; \{m,\, a,\, r,\, e\},\;\; \{m,\, a,\, s,\, t,\, e,\, r\},\;\; \{m,\, e\},\;\; \{r,\, a,\, m\}$$

has no transversal, by showing explicitly that Hall's condition is not satisfied.

4. Let $\{X_1,\, X_2,\, \ldots,\, X_n\}$ be a family of sets and let X denote their union. Show that if the family has a transversal then, given any $x \in X$, there is a transversal containing x.

Review Section 17.7

1. Show that if a 3-regular graph has a Hamiltonian cycle, then it has an edge 3-coloring.

2. There are n married couples at a party. Conversations only take place between two people of the opposite sex who are not married. Represent this situation by means of a bipartite graph and show explicitly that your graph has an edge $(n-1)$-coloring.

3. Let $S = \{a, d, i, m, o, r, s, t\}$ and let $\mathcal{S}$ be the family of subsets

$$\mathcal{S} = \{\{r, o, a, d\}, \{r, i, o, t\}, \{r, i, d, s\}, \{s, t, a, r\}, \{m, o, a, t\}, \{d, a, m, s\}, \{m, i, s, t\}\}.$$

Show that any 7-subset of S is a transversal for $\mathcal{S}$.

4. Let X denote the union of the family of sets $\{X_1, X_2, \ldots, X_n\}$, and suppose x and y are members of X. Show by an example that the family may have a transversal, but not a transversal which contains both x and y.

5. Let the elements of $\mathbb{Z}_{14}$ represent the vertices of the cycle graph C_{14}, and let G be the graph obtained from C_{14} by adding the edges $\{i, i+5\}$, $i = 0, 2, 4, 6, 8, 10, 12$. Show that G is bipartite and construct an edge coloring which uses the smallest possible number of colors. (NOTE: G is called *Heawood's graph.*)

6. Suppose there are five committees:

$$C_1 = \{a, c, e\}, \quad C_2 = \{b, c\}, \quad C_3 = \{a, b, d\}, \quad C_4 = \{d, e, f\}, \quad C_5 = \{e, f\}.$$

Each committee must send a different representative to the Annual Congress of Committees and C_1 wishes to nominate e, C_2 wishes to nominate b, C_3 wishes to nominate a and C_4 wishes to nominate f.

(i) Show that it is not possible to respect the wishes of C_1, C_2, C_3 and C_4.
(ii) Use the augmenting path method to find a complete system of distinct representatives.
(iii) Is it possible to find a complete system of distinct representatives if committee C_1 refuses to change its nomination ?

9. Prove that the complete graph K_{2m} has an edge $(2m-1)$-coloring.

10. In *projective geometry* two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are said to be *in perspective* if A_1A_2, B_1B_2 and C_1C_2 have a common point X. Desargues' theorem asserts that if the triangles are in perspective then the points $P =$ intersection of A_1B_1 and A_2B_2, $Q =$ intersection of B_1C_1 and B_2C_2, and $R =$ intersection of A_1C_1 and A_2C_2, are collinear.

Let G be the bipartite graph whose vertices represent the ten points and ten lines occurring in the theorem, two vertices being adjacent if and only if they represent a point on the corresponding line. Show that G has a Hamiltonian cycle and construct an edge 3-coloring of G.

13. Show that the following infinite family of sets satisfies Hall's condition but does not have a transversal:

$$X_0 = \{1, 2, 3, \dots\}, \;\; X_1 = \{1\}, \;\; X_2 = \{1, 2\}, \; \dots \; X_i = \{1, 2, \dots, i\}, \; \dots$$

14. Let X be the union of the family of sets $X_1, X_2, \dots, X_n$, and suppose that the family has a transversal. Show that there is a unique transversal if and only if $|X| = n$.

15. Suppose that we are given two partitions of a set X:

$$X = A_1 \cup A_2 \cup \cdots \cup A_n = B_1 \cup B_2 \cup \cdots \cup B_n.$$

A *simoultaneous transversal* is a set $\{x_1, x_2, \dots, x_n\}$ of distinct elements of X such that each part of either partition contains one x_i. Prove that there is a simoultaneous transversal if and only if no k of the parts A_i are contained in fewer than k of the parts B_j $(1 \le k \le n-1)$.

16. Let m and n be positive integers such that $m \geq n$. Construct an explicit edge m-coloring of $K_{m,n}$.

17. Show that if a k-regular graph has an odd number of vertices, then it cannot have an edge k-coloring.

18. Suppose that G is a graph with n vertices, m edges and maximum degree K. Show that if $m > K\lfloor n/2 \rfloor$ then G does not have an edge K-coloring.

19. A *vertex cover* in a graph G is a set of vertices C such that every edge of G contains at least one vertex in C. Show that if G is bipartite then the size of a maximum matching is equal to the size of a minimum vertex cover.