ANSWERS: CHAPTER 17

Section 17.1 (Relations and bipartite graphs)

1. See Figure 17.1.1. We have

$$\sum_{x \in X} \deg(x) = 2+2+1+0+1 = \sum_{y \in Y} \deg(y) = 2+2+0+2+0 = 6 = |E|.$$

2. **(i)** $\deg(x) = s$ for all $x \in X$.
(ii) $\deg(y) = r$ for all $y \in Y$.
(iii) $|E(K_{r,s})| = rs$.
(iv) A group of r desperate, recently graduated Ph.D.s in math, all apply for each of s available low-paying postdocs (here $r > s$ usually !). An edge indicates that the respective person is interested in the respective job.
(v) There are obviously no cycles, since every edge is incident to the same vertex on one side of the graph.
(vi) Since $K_{r,s}$ contains $K_{t,u}$ as a subgraph whenever $r \geq t$ and $s \geq u$, it suffices to show that $K_{2,2}$ is not a tree. But $K_{2,2} \cong C_4$.

3. No, since it has 7-cycles, for example

$$a \to b \to e \to f \to h \to g \to d \to a.$$

Section 17.2 (Edge colorings of graphs)

1. **(i)** K_4 is regular of degree 3, so we certainly need at least 3 colors. If we label the vertices 1, 2, 3, 4, then an example of an edge 3-coloring is

$$\{\{1,2\},\{3,4\}\}, \ \{\{1,3\},\{2,4\}\}, \ \{\{1,4\},\{2,3\}\}.$$

(ii) Any matching contains at most $\lfloor 5/2 \rfloor = 2$ edges. There are $\binom{5}{2} = 10$ edges in all, hence any edge coloring requires at least $10/2 = 5$ colors. If we label the vertices 1, 2, 3, 4, 5 then an example of an edge 5-coloring is

$$\{\{1,2\},\{3,5\}\}, \ \{\{1,3\},\{4,5\}\}, \ \{\{1,4\},\{2,3\}\}, \ \{\{1,5\},\{2,4\}\}, \ \{\{2,5\},\{3,4\}\}.$$

(iii) Q_3 is regular of degree 3, so we need at least 3 colors. If we label the $2^3 = 8$ vertices with all binary strings of length 3, then an explicit edge 3-coloring is gotten by using color i on an edge where the endpoints differ in coordinate i, for $i = 1, 2, 3$.

2. Consider the usual pentagonal representation of Petersen's graph (Figure 13.1(i) in the lecture notes) and suppose we try to edge color with colors A, B, C. Up to a rotation of the figure and permutation of the colors, the only way to color the outer pentagon is A, B, A, B, C. Then the five "spokes" will be colored clockwise as C, C, C, A, B. But then the inner vertex incident to the spoke colored B will be adjacent to two other inner vertices indicent to spokes colored C. Thus two incident edges would get the same color A, contradiction.

3. This follows immediately from Theorem 17.2 in the book. We can also describe an explicit edge n-coloring. Write $K_{n,n} = (X, Y, E)$, where $X = Y = \{1, 2, \ldots, n\}$ and $E = X \times Y$. Use color i, $1 \leq i \leq n$, on precisely those edges (x, y) such that $y \equiv x + i - 1 \pmod{n}$.

4. The graph has bipartition $G = (X, Y, E)$ where $X = \{1, 3, 5, 7, 10, 12, 14, 16\}$, $Y = \{2, 4, 6, 8, 9, 11, 13, 15\}$. An example of an edge 3-coloring with colors A, B, C is: use colors A and B alternately on the outer 8-cycle, color every "spoke" C and use A and B alternately again on the inner 8-cycle.

5. The idea from Exercise 17.2.1(iii) can be generalised: use color i on every edge such that the two incident vertices differ in the i:th coordinate.

Section 17.4 (Matchings)

1. Let $A = \{x_1, x_2, x_4\}$. Then $N(A) = \{y_2, y_4\}$ so $|N(A)| = 2 < 3 = |A|$.

2. **(i)** An example of such a path is $x_2 \to y_5 \rightsquigarrow x_5 \to y_3$.

(ii) $M' = \{\{x_2, y_5\}, \{x_5, y_3\}, \{x_3, y_2\}, \{x_4, y_4\}\}$.

(v) Yes, since a matching of size 5 would be a perfect matching and Hall's condition is not satisfied, as shown in Ex. 17.4.1.

3. Let $G = (X, Y, E)$ be the bipartite graph such that X is the set of people, Y the set of books and an edge represents a book that a person wishes to borrow. We are told that this graph is regular of degree k and what we're asked to prove is that it has a perfect matching. This in fact follows immediately from Theorem 17.2 in Biggs (which is proven by the same method as Hall's theorem), but we can also verify directly that Hall's condition is satisfied. Let $A \subseteq X$. Since every vertex has degree k we have that $\sum_{x \in A} \deg(x) = k \cdot |A|$. But every edge counted in the sum is incident to a vertex in $N(A)$. Hence the sum is at most $\sum_{y \in N(A)} \deg(y)$. But every vertex in Y has degree k also, hence we conclude that

$$k|A| = \sum_{x \in A} \deg(x) \leq \sum_{y \in N(A)} \deg(y) = k|N(A)| \quad \Rightarrow \quad |A| \leq |N(A)|, \text{ v.s.v.}$$

Section 17.5 (Maximum matchings)

1. An augmenting path for M is $d \to y \rightsquigarrow c \to w$. Hence, we first augment to

$$M' = (M \cup \{dy,\, cw\}) \backslash \{cy\} = \{av,\, bz,\, cw,\, dy\}.$$

An augmenting path for M' is $e \to z \rightsquigarrow b \to v \rightsquigarrow a \to x$, so we now augment to the perfect matching M'' given by

$$M'' = (M' \cup \{ez,\, bv,\, ax\}) \backslash \{bz,\, av\} = \{ax,\, bv,\, cw,\, dy,\, ez\}.$$

2. $N(\{x_1,\, x_2,\, x_3\}) = \{y_2,\, y_3\}$, so these three x's cannot be all matched. Every other 3-subset of X can be matched, for example as follows:

$$(x_1,\, x_2,\, x_4) \mapsto (y_2,\, y_3,\, y_1), \quad (x_1,\, x_3,\, x_4) \mapsto (y_2,\, y_3,\, y_4), \quad (x_2,\, x_3,\, x_4) \mapsto (y_2,\, y_3,\, y_4).$$

3. See Homework 2.

Section 17.6 (Transversals for families of finite sets)

1. For example: $a,\, l,\, b,\, e,\, t,\, s$.

2. If so, then we must choose a for the fourth set, and then we must in turn choose t for the fifth. But then $s,\, a,\, l,\, t$ have all been chosen already and so we're screwed for the sixth set !

3. If we consider the 5 words other than "master", then they contain a total of only 4 different letters, namely $a,\, e,\, m,\, r$.

4. Suppose there is a transversal not containing x. Since $x \in \cup_{i=1}^{n} X_i$, there must be some i such that $x \in X_i$. Let x_i be the element of X_i appearing in the transveral. Then swapping x_i for x yields a new transversal, containing x.

Review Section 17.7

1. Since G is 3-regular, if we remove a Hamiltonian cycle from it then we'll be left with something 1-regular, i.e.: a perfect matching. In particular, this means G has an even number of nodes. Hence, the Hamiltonian cycle has even length and can be edge 2-colored, and then the third color can be used on the edges in the remaining matching.

2. Let $G = (X,\, Y,\, E)$ be the bipartite graph where X is the set of mean at the party and Y the set of women. Index both sets with $1,\, 2,\, \ldots,\, n$, so that a married couple get the same index. Then E consists of all pairs $(x,\, y)$ such that $x \neq y$.

An edge $(n-1)$-coloring is gotten by coloring the edge $(x,\, y)$ with color i if and only if $y \equiv x + i \pmod{n}$.

3. The following table gives a transversal for each 7-element subset of S. The letters in the transversal represent the words in the left-to-right order in which they are listed in the exercise.

Letter not used	Transversal
a	r, i, d, s, o, m, t
d	r, i, s, a, o, m, t
i	r, o, s, a, m, d, t
m	r, o, s, a, t, d, i
o	r, i, s, a, t, d, m
r	o, i, s, a, t, d, m
s	r, o, d, a, t, m, i
t	r, o, s, a, m, d, i

4. Take $n = 2$, $X_1 = \{1, 2\}$, $X_2 = \{3\}$. Then either $\{1, 3\}$ or $\{2, 3\}$ is a transveral, but not $\{1, 2\}$.

5. Addition below is to be interpreted as modulo 14. We have $G = (X, Y, E)$ where $X = \{2k : 0 \leq k \leq 6\}$ and $Y = \{x + 1 : x \in X\}$. G is 3-regular, so any edge coloring must use at least 3 colors. But the edges $(i, i+1)$ form a 14-cycle and the edges $(i, i+5)$ form a perfect matching of size 7. Hence we can 2-color the cycle and use the third color on the matching. So $\Phi(G) = 3$ (note that this also follows from Theorem 17.2 in Biggs).

6. **(i)** If we accomodated each C_i, $i = 1, \ldots, 4$, then each of e and f would already have been nominated and there'd be no option left for C_5.
(ii) It's easier to just do it by inspection, e.g.: a, b, d, e, f.
(iii) Yes: e, c, b, d, f.
9.

10.

13.

14.

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