EXERCISES: CHAPTER 18

Section 18.1 (Digraphs)

1. In the adjacency list for a digraph we place y in column x whenever $(x,\, y)$ is an arc. Sketch the digraph whose adjacency list is

a	b	c	d	e	f
d	a	b	b	f	a
e			c		
			e		

Find a directed path from c to f, and a directed cycle starting and ending at d.

2. In the following table, there is a $+$ in row i and column j if i beats j, and a $-$ if j beats i. Find a directed path containing all the vertices of the tournament.

	1	2	3	4	5	6	7	8	9
1		$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$
2			$-$	$+$	$+$	$-$	$+$	$-$	$-$
3				$+$	$-$	$-$	$+$	$-$	$+$
4					$+$	$-$	$-$	$+$	$-$
5						$-$	$+$	$-$	$-$
6							$+$	$-$	$-$
7								$+$	$+$
8									$-$

4. Prove that, for any digraph $G = (V,\, E)$,

$$\sum_{v \in V} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v).$$

Prove that, if G is a tournament, then

$$\sum_{v \in V} [\text{indeg}(v)]^2 = \sum_{v \in V} [\text{outdeg}(v)]^2.$$

Section 18.3 (Flows and cuts)

1. Find a flow f^* in Figure 18.3.1 for which $|f^*| = 10$. Why is this the maximum possible value ?

2. Sketch the network whose vertices are s, a, b, c, d, t and whose arcs and capacities are

Arc	(s, a)	(s, b)	(a, b)	(a, c)	(b, d)	(d, c)	(c, t)	(d, t)
Capacity	5	3	3	3	5	2	6	2

Find a flow with value 7 and a cut with capacity 7. What is the maximum value of a flow, and why ?

3. Let $D = (V, A)$ be a digraph and suppose $\phi : A \to \mathbb{N}$ is any function, not necessarily a flow. Show that

$$\sum_{v \in V} \text{outflow}(v) = \sum_{v \in V} \text{inflow}(v).$$

Deduce that if s and t are the source and sink of a network, and ϕ is a flow, then

$$\text{outflow}(s) = \text{inflow}(t).$$

Section 18.4 (The max-flow min-cut theorem)

1. The diagram in Figure 18.4.1 represents a network and the numbers on the arcs are their capacities. A flow f is defined as follows:

Arc	(s, a)	(s, b)	(s, c)	(a, b)	(a, d)	(b, c)	(b, d)	(b, e)	(c, e)	(d, t)	(e, t)
Flow	5	6	0	0	5	1	2	3	1	7	4

(i) What is the value of f ?
(ii) Find an f-augmenting path and compute the value of the augmented flow.
(iii) Find a cut with capacity 12.
(iv) What can you deduce ?

2. The vertex set of a network is $\{s, a, b, c, d, e, t\}$ and the capacity of an arc (x, y), if there is such an arc, is given in row x and column y of the following table. Find a maximum flow from s to t in the network and explain why it is a maximum.

	s	a	b	c	d	e	t
s	$-$	14	$-$	14	$-$	12	$-$
a	$-$	$-$	8	14	$-$	3	$-$
b	$-$	$-$	$-$	$-$	$-$	2	15
c	$-$	$-$	$-$	$-$	19	$-$	$-$
d	$-$	$-$	6	$-$	$-$	1	10
e	$-$	$-$	$-$	$-$	$-$	$-$	14

Section 18.5 (The labelling algorithm for network flows)

1. Starting from the zero flow, use the Ford-Fulkerson algorithm to find the maximum flow in the network illustrated in Figure 18.5.1.

2. A commodity produced at s_1 and s_2 is transported through the network illustrated in Figure 18.5.2 to markets at t_1, t_2 and t_3.
(i) Add a new "supersource" and "supersink" to make a network of the standard kind.
(ii) Find a "good" initial flow by inspection.
(iii) Use the Ford-Fulkerson algorithm to find a maximum flow.

Review Section 18.6

1. Find a directed path containing all the vertices of the tournament described by the following table:

	1	2	3	4	5	6	7	8
1		$+$	$-$	$+$	$-$	$-$	$-$	$+$
2			$+$	$-$	$-$	$+$	$-$	$+$
3				$-$	$-$	$+$	$+$	$+$
4					$+$	$+$	$+$	$+$
5						$-$	$-$	$-$
6							$+$	$-$
7								$+$

2. Define the *score* of a vertex in a tournament to be its outdegree, and the *score sequence* of the tournament to be the sequence of all the scores arranged in non-decreasing order.
(i) What is the score sequence of the tournament in Exercise 18.1.2 ?
(ii) Show that, in general, if the score sequence is $(s_1, s_2, \ldots, s_n)$ then

$$\sum_{i=1}^{n} s_i = \frac{n(n-1)}{2}.$$

3. Show that the score sequence of a tournament on n vertices always satisfies

(i) $\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2}$, for each $k = 1, \ldots, n-1$.
(ii) $\frac{k-1}{2} \leq s_k \leq \frac{n+k-2}{2}$, for each $k = 1, \ldots, n$.

4. A tournament is said to be *transitive* if it represents a transitive relation: that is, if the existence of the arcs (x, y) and (y, z) implies that of the arc (x, z). Show that a tournament is transitive if and only if the k:th term of the score sequence is $k-1$, for every $k = 1, \ldots, n$.

5. What is the number of tournaments on n vertices ? How many of them are transitive ?

6. Show that in any tournament there is a vertex s such that any vertex x can be reached from s by a directed path of length 0, 1 or 2.

9. Consider the network with vertex set $\{s, a, b, c, t\}$ and arcs and capacities given by the following table:

Arc	(s, a)	(s, b)	(a, b)	(a, c)	(a, t)	(b, c)	(c, t)
Capacity	5	2	3	1	3	3	4

Calculate the capacities of all cuts separating s from t, and find a maximum flow from s to t.

10. Use the Ford-Fulkerson algorithm to verify that if the capacities of all the arcs in a network are integers, then there is a maximum flow such that the flow on each arc is an integer.

11. Show that the maximum flow in Figure 18.5.2 is not uniquely determined, and hence give an example of a maximum flow for which the flow on each arc is not an integer.

12. Let $(S_1, \overline{S_1})$ and $(S_2, \overline{S_2})$ be cuts separating s and t in a network, both of which are minimum cuts. Show that $(S_1 \cap S_2, \overline{S_1 \cap S_2})$ is also a minimum cut.