# ANSWERS: CHAPTER 18

## Section 18.1 (Digraphs)

**1.** See Figure 18.1.1. Two simple $c \to f$ paths are

$$c \to b \to a \to e \to f, \quad c \to b \to a \to d \to e \to f.$$

Two simple directed cycles out of $d$ are

$$d \to c \to b \to a \to d, \quad d \to e \to f \to a \to d.$$

**2.** For example,

$$1 \to 2 \to 4 \to 5 \to 7 \to 9 \to 8 \to 6 \to 3.$$

**4.** **(i)** Every edge in the graph is counted exactly once in each sum, since each comes *out* from exactly one vertex and goes *in* to exactly one other. In other words, both sums equal $|E|$.
**(ii)** To simplify notation, denote the indegree (resp. outdegree) of vertex $v$ by $i_v$ (resp. $o_v$). Since $G$ is a tournament, $i_v + o_v = n - 1$ for every $v \in V$, where $|V| = n$. Note also from part (i) that $\sum_{v\in V} i_v = \sum_{v\in V} o_v = |E| = \frac{n(n-1)}{2}$. Hence,

$$\sum_{v\in V} i_v^2 = \sum_{v\in V}[(n-1) - o_v]^2 = \sum_v (n-1)^2 + \sum_v o_v^2 - \sum_v 2(n-1)o_v =$$

$$= (n-1)^2 \sum_v 1 + \sum_v o_v^2 - 2(n-1)\sum_v o_v =$$

$$= (n-1)^2|V| + \sum_v o_v^2 - 2(n-1)|E| =$$

$$= n(n-1)^2 + \sum_v o_v^2 - 2(n-1)\left[\frac{n(n-1)}{2}\right] = \sum_v o_v^2, \quad \text{v.s.v.}$$

## Section 18.3 (Flows and cuts)

**1.** See Figure 18.3.1 in the facit for a flow of value 10. This is maximum since if we take $\mathcal{S} = \{s, a, b, d\}$, $\mathcal{T} = \{c, t\}$, then the capacity of the cut $(\mathcal{S}, \mathcal{T})$ is

$$c(\mathcal{S}, \mathcal{T}) = c(a, t) + c(d, t) + c(s, c) = 3 + 4 + 3 = 10.$$

**2.** See Figure 18.3.2 in the facit for the network and a flow of value 7. A cut of capacity 7 is given by $\mathcal{S} = \{s, a, b\}$, $\mathcal{T} = \{c, d, t\}$, since

$$c(\mathcal{S}, \mathcal{T}) = c(a, c) + c(b, d) = 3 + 4 = 7.$$

By the max-flow min-cut theorem, 7 is the maximum value of a flow.

**3.** **(i)** For any directed edge $(v, w)$, the flow along the edge flows *out* from $v$ and *in* to $w$. Hence, both sums are just the sum of the flows along all edges in the network, i.e.: both sums equal $\sum_{e\in A} \phi(e)$.
**(ii)** If $\phi$ is a flow then, for any vertex $v$ which is neither the source nor the sink one has, by conservation of flow, $\text{outflow}(v) = \text{inflow}(v)$. Hence, from part (i) it follows in this case, after cancellation of all terms involving $v \notin \{s, t\}$, that

$$\text{outflow}(s) + \text{outflow}(t) = \text{inflow}(s) + \text{inflow}(t).$$

But there is no inflow to a source and no outflow from a sink, hence the above reduces to $\text{outflow}(s) = \text{inflow}(t)$, v.s.v.

## Section 18.4 (The max-flow min-cut theorem)

**1.** **(i)** The value of $f$ is the total flow out of the source $s$, namely $5 + 6 + 0 = 11$.
**(ii)** For example, $s \to c \to e \to b \to d \to t$. We can increase the flow by one along this path. Note that in the case of the "backwards" arc $(b, e)$, this means reducing the flow along it by one.
**(iii)** If we augment the flow along the path identified in (ii), then for the new flow $f^*$, the set of vertices reachable from $s$ by an $f^*$-augmenting path is $\mathcal{S} = \{s, b, c, e\}$, while the set of unreachable ones is $\mathcal{T} = \{a, d, t\}$. We have

$$c(\mathcal{S}, \mathcal{T}) = c(s, a) + c(b, d) + c(e, t) = 5 + 3 + 4 = 12.$$

**(iv)** The flow $f^*$, which has value 12, must be maximum.

**2.** See Figure 18.4.2 in the facit for both the network and a flow $f$ of value 38. For this flow, the set of vertices reachable from $s$ by an $f$-augmenting path is $\mathcal{S} = \{s, a, c, d, e\}$, while the set of unreachable ones is $\mathcal{T} = \{b, t\}$. We have

$$c(\mathcal{S}, \mathcal{T}) = c(a, b) + c(d, b) + c(d, t) + c(e, t) = 8 + 6 + 10 + 14 = 38.$$

Hence, the flow $f$ must be maximum.

## Section 18.5 (The labelling algorithm for network flows)

**1.** We can find the following sequence of augmenting paths:

| Step | Augmenting path | Increase in flow |
|---|---|---|
| 1 | $s \to a \to e \to t$ | 40 |
| 2 | $s \to a \to c \to t$ | 10 |
| 3 | $s \to b \to c \to t$ | 5 |

At this point, each of the two edges into $t$ is saturated, so the flow must be maximum. This maximum flow is illustrated in Figure 18.5.1 in the facit.

**2.** **(i)** We add a supersource $s$ to the left of $s_1$ and $s_2$, and an arc from $s$ to each of

$s_1$ and $s_2$ of infinite capacity (or at least of sufficiently large capacity that these arcs can never become saturated by a maximum flow). Similarly, we place a supersink $t$ to the right of the three $t_i$'s and an infinite-capacity arc from each $t_i$ to it.
**(ii), (iii)** The initial flow $f_0$ which I found "by inspection" is illustrated in Figure 18.5.2(i) in the facit. We can augment the flow as follows. Squiggles represent "backwards" edges where the flow is reduced.

| Step | Augmenting path | Increase in flow |
|---|---|---|
| 1 | $s \to s_1 \to b \to t_3$ | 2 |
| 2 | $s \to s_1 \to b \to c \rightsquigarrow a \rightsquigarrow d \to t_1$ | 2 |

The flow $f$ at this point is illustrated in Figure 18.5.2(ii) in the facit. It has value $f(s,\, s_1) + f(s,\, s_2) = 27 + 12 = 39$. The set of vertices reachable from $s$ by an $f$-augmenting path is $\mathcal{S} = \{s,\, s_1,\, s_2,\, a,\, b,\, c,\, d\}$, while the set of unreachable ones is $\mathcal{T} = \{e,\, t_1,\, t_2,\, t_3,\, t\}$. We have

$$c(\mathcal{S},\, \mathcal{T}) = c(a,\, t_1)+c(d,\, t_1)+c(d,\, t_2)+c(c,\, e)+c(b,\, t_3)+c(s_2,\, e) = 4+2+7+16+4+6 = 39.$$

Hence, the flow $f$ must be maximum.

## Review Section 18.6

**1.** For example,

$$1 \to 4 \to 2 \to 3 \to 6 \to 7 \to 8 \to 5.$$

**2. (i)** The scores of the 9 players, ordered $1-9$, are $4,\, 3,\, 4,\, 3,\, 3,\, 5,\, 4,\, 4,\, 6$. Hence the score sequence is $(3,\, 3,\, 3,\, 4,\, 4,\, 4,\, 4,\, 5,\, 6)$.
**(ii)** Every match has a single winner. Hence the sum of the scores is just the total number of matches played which, since everyone plays everyone once, is $\binom{n}{2} = \frac{n(n-1)}{2}$, v.s.v.

**3. (i)** Every match between two of the $k$ worst players contributes to the score of one of them. Hence the sum of their scores must be at least $\binom{k}{2}$.
**(ii)** LHS: From part (i) and the fact that the $s_i$ are non-decreasing, we have

$$ks_k \geq \sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2} \quad \Rightarrow \quad s_k \geq \frac{k-1}{2}, \quad \text{v.s.v.}$$

RHS: Suppose on the contrary that $s_k > \frac{n+k-2}{2}$. Then $s_k \geq \frac{n+k-1}{2}$, since $s_k$ is an integer. Using Ex. 2, part (i) of Ex. 3 and the fact that the scores are non-decreasing it would follow that

$$\frac{n(n-1)}{2} = \sum_{i=1}^{n} s_i = \sum_{i=1}^{k-1} s_i + \sum_{i=k}^{n} s_i \geq \frac{(k-1)(k-2)}{2} + (n-k+1) \cdot \frac{n+k-1}{2}.$$

After some algebra, the right-hand side of the inequality simplifies to $\frac{n^2-(k-1)}{2}$, which is strictly greater than $\frac{n^2-n}{2}$, contradiction !

**4.** It's easy to see that a tournament on $n$ players is transitive if and only if the players can be labelled from 1 to $n$ such that $i$ beats $j$ if and only if $i < j$. Thus player $k$ beats those from 1 through $k-1$ and no-one else, i.e.: $s_k = k-1$, v.s.v.

**5.** There are $2^{\binom{n}{2}}$ tournaments since we have two choices for the direction of each of the $\binom{n}{2}$ edges of $K_n$. A transitive tournament is determined by a permutation of the vertices, so there are $n!$ of them[1].

**6.** Pick a player $s$ with the highest score. Let $L$ (resp. $W$) be the set of players to whom $s$ loses (resp. against whom $s$ wins). If $x = s$, then there is a path of length zero from $s$ to $x$, duh ! If $x \in W$, then $s \to x$ is a path of length one in the tournament. Finally, suppose $x \in L$. If there is no path of length two from $s$ to $x$ then in particular it means that, for every $y \in W$, $s \to y \to x$ is not a path in the tournament. In other words, $x$ must have beaten every $y \in W$, i.e.: $x$ beat everyone whom $s$ beat. But $x$ also beat $s$, contradicting the assumption that $s$ had the highest score.

**9.** First, let's think about how many cuts there are. A cut is a partition $(\mathcal{S}, \mathcal{T})$ of $\{s,\, a,\, b,\, c,\, t\}$ such that $s \in \mathcal{S}$ and $t \in \mathcal{T}$. Hence what determines a cut is which subset of $\{a,\, b,\, c\}$ one puts in $\mathcal{S}$. Hence there are $2^3 = 8$ possible cuts. The full list of them, along with their capacities, is given in the following table:

| $\mathcal{S}$ | $\mathcal{T}$ | $c(\mathcal{S},\, \mathcal{T})$ |
|---|---|---|
| $\{s\}$ | $\{a,\, b,\, c,\, t\}$ | $c(s,\, a) + c(s,\, b) = 5 + 2 = 7$ |
| $\{s,\, a\}$ | $\{b,\, c,\, t\}$ | $c(s,\, b) + c(a,\, b) + c(a,\, c) + c(a,\, t) = 2 + 3 + 1 + 3 = 9$ |
| $\{s,\, b\}$ | $\{a,\, c,\, t\}$ | $c(s,\, a) + c(b,\, c) = 5 + 3 = 8$ |
| $\{s,\, c\}$ | $\{a,\, b,\, t\}$ | $c(s,\, a) + c(s,\, b) + c(c,\, t) = 5 + 2 + 4 = 11$ |
| $\{s,\, a,\, b\}$ | $\{c,\, t\}$ | $c(a,\, t) + c(a,\, c) + c(b,\, c) = 3 + 1 + 3 = 7$ |
| $\{s,\, a,\, c\}$ | $\{b,\, t\}$ | $c(s,\, b) + c(a,\, b) + c(a,\, t) + c(c,\, t) = 2 + 3 + 3 + 4 = 12$ |
| $\{s,\, b,\, c\}$ | $\{a,\, t\}$ | $c(s,\, a) + c(c,\, t) = 5 + 4 = 9$ |
| $\{s,\, a,\, b,\, c\}$ | $\{t\}$ | $c(a,\, t) + c(c,\, t) = 3 + 4 = 7$ |

Since the minimum capacity of a cut is 7, this must also be the value of a maximum flow. Such a flow is given by

| Arc | $(s,\, a)$ | $(s,\, b)$ | $(a,\, b)$ | $(a,\, c)$ | $(a,\, t)$ | $(b,\, c)$ | $(c,\, t)$ |
|---|---|---|---|---|---|---|---|
| Flow | 5 | 2 | 1 | 1 | 3 | 3 | 4 |

**10.** Suppose we apply the algorithm and at some point have a flow in which the flow along every edge is integer-valued. If we find an augmenting path from source to sink, then we augment the flow along the path by the minimum of the '"spare capacities" available along the edges in the path. Each spare amount is integer-valued, hence we're taking a minimum of integer amounts, which is also an integer amount. Hence, the

[1] Stirling's formula says that $n! \sim n^n e^{-n}\sqrt{2\pi n}$, so there are around $2^{n \log_2 n}$ transitive $n$-player tournaments, compared to around $2^{n^2/2}$ tournaments in all.

augmented flow will also be integer-valued. Since we can start from the everywhere-zero flow, it follows that the algorithm will always produce a maximum flow which is integer-valued everywhere.

**11.** See Figure 18.5.2(ii) in the facit for an example of a mximum flow. The simplest way to tweak this and still have a maximum flow is to add one unit to the arc $(s_2,\, a)$, which has four units of spare capacity, and subtract one unit from the arc $(s_1,\, a)$. Call the old flow $f_1$ and the new flow $f_2$. Now note that $f = \frac{f_1+f_2}{2}$ is also a maximum flow[2], by which we mean that the flow along each arc in $f$ is the average of the corrsponding flows in $f_1$ and $f_2$. But $f$ will be half-integer valued along the arcs $(s_1,\, a)$ and $(s_2,\, a)$.

**12.** I claim that, in fact, for any two cuts, one has

$$c(S_1,\, \overline{S_1}) + c(S_2,\, \overline{S_2}) \geq c(S_1 \cap S_2, \overline{S_1 \cap S_2}) + c(S_1 \cup S_2,\, \overline{S_1 \cup S_2}). \tag{0.1}$$

In particular, if both the cuts on the left of (0.1) are minimal, then so must be both cuts on the right, and we're done. Moreover, we must have equality in (0.1) in that case.

To prove (0.1) in general, recall that the capacity of a cut is, by definition, the sum of the capacities of the arcs that cross it in the "right" direction. Hence it suffices to prove that, for any arc $(v,\, w)$ in the network, it appears at least as many times on the left of (0.1) as it does on the right. We consider four cases:

*Case 1:* $v \notin S_1 \cup S_2$. Then, for any $w$, the arc $(v,\, w)$ will not appear at all on either side of (0.1).
*Case 2:* $v \in S_1 \backslash S_2$. The arc $(v,\, w)$ will appear once on the left of (0.1), namely in the sum for $c(S_1,\, \overline{S_1})$, if and only if $w \notin S_1$, otherwise it will not appear at all. On the right, since $v \notin S_2$, the arc $(v,\, w)$ will never appear in the sum for $c(S_1 \cap S_2,\, \overline{S_1 \cap S_2})$. It will appear in the sum for $c(S_1 \cup S_2,\, \overline{S_1 \cup S_2})$ if and only if $w \notin S_1 \cup S_2$. If that holds, then a fortiori $w \notin S_1$ holds, so if the arc appears on the right it must also do so on the left.
*Case 3:* $v \in S_2 \backslash S_1$. This case is completely analogous to Case 2, just interchange the roles of $S_1$ and $S_2$.
*Case 4:* $v \in S_1 \cap S_2$. On the left of (0.1), the arc $(v,\, w)$ will appear
- zero times if $w \notin \overline{S_1} \cup \overline{S_2}$, i.e.: if $w \in S_1 \cap S_2$,
- once if $w \in S_1 \Delta S_2$,
- twice if $w \in \overline{S_1} \cap \overline{S_2}$, i.e. if $w \in \overline{S_1 \cup S_2}$.

It's easy to check that exactly the same is true on the right-hand side of (0.1). This completes the proof of (0.1).

[2] More generally, any *convex combination* of maximum flows is also a maximum flow, i.e.: if $f_1,\, f_2,\, \ldots,\, f_n$ are all maximum flows and $\lambda_1,\, \lambda_2, \ldots,\, \lambda_n$ are non-negative constants satisfying $\sum_{i=1}^{n} \lambda_i = 1$, then $\sum_{i=1}^{n} \lambda_i f_i$ is also a mximum flow.