EXERCISES: CHAPTER 19

Section 19.1 (Generalities about recursion)

1. Show that the formula (3.10) for d_n can be written as

$$d_n = \sum_{k=2}^{n} (-1)^k \times \prod_{j=k+1}^{n} j.$$

Show that the number of multiplications required to compute d_n by this formula is $O(n^2)$[1]. What is the number of multiplications required if one uses the recursion in Ex. 11.4.4 ?

3. Show that the sequence (d_n) also satisfies the following recursion:

$$d_1 = 0, \quad d_n = nd_{n-1} + (-1)^n \;\; \forall\, n \geq 2.$$

Is there any advantage in using this recursion rather than the usual one (from Ex. 11.4.4) for the calculation of d_n ?

Section 19.2 (Linear recursion)

1. Find an explicit formula for u_n when

(i) $u_0 = 1$, $u_1 = 1$, $u_{n+2} - 3u_{n+1} - 4u_n = 0 \; \forall\, n \geq 0$
(ii) $u_0 = -2$, $u_1 = 1$, $u_{n+2} - 2u_{n+1} + u_n = 0 \; \forall\, n \geq 0$.

4. Let q_n denote the number of words of length n in the alphabet $\{0, 1\}$ which have the property that no two consecutive terms are 0. What is the relationship between the numbers q_n and the Fibonacci numbers f_n discussed in class ?

5. Without using the formula (4.7) for the Fibonacci numbers f_n, prove that

(i) $f_{n+2} = 1 + \sum_{k=1}^{n} f_k$.
(ii) $f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1}$.

[1]That is, there is an absolute constant $C > 0$ such that the number of multiplications required to compute d_n is at most Cn^2, for all $n \in \mathbb{N}_0$.

Review Section 19.7

1. Find explicit formulae for the terms of the sequences defined by

(i) $u_0 = 0$, $u_1 = 1$, $u_{n+2} + u_{n+1} - 2u_n = 0 \; \forall \; n \geq 0$
(ii) $u_0 = 1$, $u_1 = 0$, $u_{n+2} - 6u_{n+1} + 8u_n = 0 \; \forall \; n \geq 0$.

2. Show that the equation

$$n(n+1)u_{n+2} - 5n(n+2)u_{n+1} + 4(n+1)(n+2)u_n = 0$$

is satisfied by $u_n = n$. Use the substitution $u_n = nv_n$ to show that the solution for which $u_1 = 12$ and $u_2 = 60$ is

$$u_n = 3n2^{2n-1} + 6n.$$

3. Find a formula for the nth term of the sequence (u_n) defined by

$$u_0 = X, \quad u_1 = Y, \quad u_{n+2} = u_n + n \;\; \forall \; n \geq 0.$$

4. An ordered triple (a, b, c) of integers is said to be an *(increasing, non-trivial) arithmetic progression (AP)* if $a < b < c$ and $c - b = b - a$. Let L_n denote the number of APs whose elements belong to $\mathbb{N}_n$. Show that $L_{2n+1} = L_{2n} + n$ and derive a similar equation for L_{2n}. Deduce that L_n satisfies the same recursion as in Ex. 3 and find a formula for L_n.

5. Let C_n denote the cycle graph with n vertices and let $f_n(k)$ be the number of vertex colourings of C_n when there are k colours available. By splitting the set of colourings into two parts, according to whether vertices 0 and 2 receive the same colour or not, show that

$$f_n(k) = (k-1)f_{n-2}(k) + (k-2)f_{n-1}(k) \;\; \forall \; n \geq 5.$$

Deduce that

$$f_n(k) = (k-1)[(k-1)^{n-1} + (-1)^n] \;\; \forall \; n \geq 3.$$

6. Show that the number of vertex colourings of any tree with n vertices, when there are k colours available, is $k(k-1)^{n-1}$.

10. Let (f_n) denote the Fibonacci numbers. Show that

(i) $\sum_{k=1}^{n} f_{2k} = f_{2n+1} - 1$
(ii) $f_{n+1}^3 + f_n^3 - f_{n-1}^3 = f_{3n}$.

11. Let $\lambda(n, k)$ denote the number of k-element subsets of $\mathbb{N}_n$ which do not contain two consecutive integers. Show that

$$\lambda(n, k) = \lambda(n-2, k-1) + \lambda(n-1, k)$$

and hence verify that

$$\lambda(n, k) = \binom{n-k+1}{k}.$$

Can you prove this formula without using the recurrence relation ? (HINT: Example 2.11).

12. Let $\mu(n,\,k)$ denote the number of ways of selecting k objects from n objects arranged in a circle, in such a way that no two are adjacent. Show that if $\lambda(n,\,k)$ is as in the previous exercise, then

$$\mu(n,\,k) = \lambda(n-1,\,k) + \lambda(n-3,\,k-1).$$

Deduce that

$$\mu(n,\,k) = \frac{n}{n-k}\binom{n-k}{k}.$$