ANSWERS: CHAPTER 19

Section 19.1 (Generalities about recursion)

1. The formula in (3.10) is, literally,

$$d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

The first two terms cancel out, leaving us with a sum from $k = 2$ to $k = n$. But for each k one has $n!/k! = \prod_{j=k+1}^{n} j$, where for $k = n$ the empty product is interpreted as being 1.

The kth term in the sum is $\pm$ a product of $n - k$ integers. Hence the total number of multiplications performed in the evaluation of d_n will be $\sum_{k=2}^{n}(n-k) = \frac{(n-1)(n-2)}{2} = O(n^2)$. **3.** It's clear that $d_1 = 0$. We can prove the recursion by induction on n, using the already known recursion from Ex. 11.4.4. For $n = 2$ we have $1 = d_2 = 2 \times 0 + 1 = 2d_1 + (-1)^2$, as desired. Suppose $d_n = nd_{n-1} + (-1)^n$ holds for some $n \geq 2$. This can be rewritten as $d_{n-1} = \frac{1}{n}(d_n - (-1)^n)$. From this and Ex. 11.4.4 we compute

$$d_{n+1} = n(d_n + d_{n-1}) = n\left[d_n + \frac{1}{n}(d_n - (-1)^n)\right] = \cdots = (n+1)d_n + (-1)^{n+1}, \quad \text{v.s.v.}$$

The advantage of using this recursion over that in Ex. 11.4.4 is that each iteration involves only one multiplication and the addition or subtraction of 1, whereas previously each iteration involved one multiplication and one "proper" addition. The latter also required a little more memory since one used both d_{n-1} and d_{n-2} to compute d_n.

Section 19.2 (Linear recursion)

1. (i) The auxiliary equation is $x^2 - 3x - 4 = 0$ which has roots $x_1 = 4$, $x_2 = -1$. The general solution to the recursion is thus

$$u_n = C_1 \cdot 4^n + C_2 \cdot (-1)^n.$$

Inserting the initial conditions $u_0 = 1$ and $u_1 = 1$ gives the two equations

$$C_1 + C_2 = 1, \quad 4C_1 - C_2 = 1,$$

whose solution is $C_1 = 2/5$, $C_2 = 3/5$. Thus

$$u_n = \frac{1}{5}\left(2 \cdot 4^n + 3 \cdot (-1)^n\right).$$

(ii) The auxiliary equation is $x^2 - 2x + 1 = 0$ which has the repeated root $x_1 = x_2 = 1$. The general solution to the recursion is thus

$$u_n = C_1 + C_2 \cdot n.$$

Inserting the initial conditions $u_0 = -2$ and $u_1 = 1$ gives $C_1 = -2$ and $C_2 = 3$. Thus $u_n = -2 + 3n$.

4. This is the same as Ex. 10.7.17, from which it follows that $q_n = f_{n+2}$ since the sequences (q_n) and (f_n) satisfy the same recurrence: $q_{n+2} = q_{n+1} + q_n$, but $q_1 = f_3 = 2$ and $q_2 = f_4 = 3$.

5. See Homework 1.

Review Section 19.7

1. (i) The auxiliary equation is $x^2 + x - 2 = 0$ which has roots $x_1 = 1$, $x_2 = -2$. The general solution to the recursion is thus

$$u_n = C_1 + C_2 \cdot (-2)^n.$$

Inserting the initial conditions $u_0 = 0$ and $u_1 = 1$ gives the two equations

$$C_1 + C_2 = 1, \quad C_1 - 2C_2 = 1,$$

whose solution is $C_1 = 1/3$, $C_2 = -1/3$. Thus

$$u_n = \frac{1}{3}\left(1 - (-2)^n\right).$$

(ii) The auxiliary equation is $x^2 - 6x + 8 = 0$ which has roots $x_1 = 4$, $x_2 = 2$. The general solution to the recursion is thus

$$u_n = C_1 \cdot 4^n + C_2 \cdot 2^n.$$

Inserting the initial conditions $u_0 = 1$ and $u_1 = 0$ gives the two equations

$$C_1 + C_2 = 1, \quad 4C_1 + 2C_2 = 0,$$

whose solution is $C_1 = -1$, $C_2 = 2$. Thus $u_n = 2^{n+1} - 4^n$.

2. Substituting $u_n = n$ we can see immediately (no need to multiply out since we have a common factor of $n(n+1)(n+2)$) that

$$n(n+1)(n+2) - 5n(n+2)(n+1) + 4(n+1)(n+2)n = 0.$$

If we instead write $u_n = nv_n$ then the recursion becomes

$$n(n+1)(n+2)v_{n+2} - 5n(n+2)(n+1)v_{n+1} + 4(n+1)(n+2)nv_n = 0.$$

Here we can cancel the common factor $n(n+1)(n+2)$ and get the linear recursion $v_{n+2} - 5v_{n+1} + 4v_n = 0$. This has auxiliary equation $x^2 - 5x + 4 = 0$, whose roots are $x_1 = 1$, $x_2 = 4$. Thus the general solution is $v_n = C_1 + C_2 \cdot 4^n$. The initial conditions $u_1 = 12$, $u_2 = 60$ correspond to $v_1 = 12$, $v_2 = 30$. Inserting these into the general solution gives the equations

$$C_1 + 4C_2 = 12, \quad C_1 + 16C_2 = 30,$$

whose solution is $C_1 = 6$, $C_2 = 3/2$. Thus $v_n = 6 + \frac{3}{2}(4^n) = 6 + 3 \cdot 2^{2n-1}$. Since $u_n = nv_n$ we finally obtain $u_n = 6n + 3n2^{2n-1}$, v.s.v.

3. Even and odd values of n don't interact here. For even indices we have

$$u_0 = X, \quad u_2 = X + 0, \quad u_4 = (X+0)+2, \quad u_6 = ((X+0)+2)+4, \ldots$$

The general pattern is clear, namely $u_{2n} = X + \sum_{k=1}^{n-1} 2k = X + n(n-1)$.

Similarly, for odd indices one can derive $u_{2n-1} = Y + \sum_{k=1}^{n-1}(2k-1) = Y + n^2$.

4. To see that $L_{2n+1} = L_{2n} + n$ we observe that, if $(a,\, b,\, c)$ is an AP in $\mathbb{N}_{2n+1}$, then either

(i) $c \le 2n$, in which case $(a,\, b,\, c)$ is an AP in $\mathbb{N}_{2n}$ and there are L_{2n} possibilities for it, or

(ii) $c = 2n+1$. In this case, since $a \ge 1$ and $c - b = b - a$, so b must be one of the numbers $n+1,\, n+2,\, \ldots,\, 2n$. So there are n possibilities for the AP in this case.

A similar argument yields $L_{2n} = L_{2n-1} + (n-1)$. Putting the two recurrences together yields, for even indices,

$$L_{2n+2} = L_{2n+1} + n = (L_{2n} + n) + n = L_{2n} + 2n,$$

and for odd indices,

$$L_{2n+1} = L_{2n} + n = (L_{2n-1} + (n-1)) + n = L_{2n-1} + (2n-1).$$

Thus, whether k is even or odd, one has $L_{k+2} = L_k + k$, as in Ex. 3. Here the initial conditions are $L_0 = L_1 = 0$, thus $L_{2n} = n(n-1)$ and $L_{2n+1} = n^2$.

5. First suppose vertices 0 and 2 get the same colour. Whatever this colour is, there will be $k-1$ options for the colour assigned to vertex 1, since it must be a different colour than that assigned to its neighbors 0 and 2. If we imagine having coloured 1 first, we can then remove it and "glue" 0 and 2 together as a single vertex, so that we are left with $n-2$ vertices round a circle which must be coloured according to the same rule as initially (neighbors get different colours). There are $f_{n-2}(k)$ possible colourings, all of which can be combined with the colour given to vertex 1. By MP, there are $(k-1)f_{n-2}(k)$ possible colourings of all n vertices in this case.

Secondly, suppose 0 and 2 get different colours. Whatever these colours are, there will be $k-2$ options for the colour given to vertex 1. Having removed vertex 1, we no longer glue together 0 and 2, since they now get different colours. We're thus left in this case with $n-1$ vertices, which can be coloured in $f_{n-1}(k)$ ways. Combining with the colour given to vertex 1 there are, by MP, a total of $(k-2)f_{n-1}(k)$ possible colourings in this case.

Finally, we obtain the desired recursion by AP.

To deduce the explicit formula for $f_n(k)$, we can proceed by induction on n for each fixed k. The base cases are $n = 3$ and $n = 4$, in which cases the formula states that

$$f_3(k) = (k-1)[(k-1)^2 + (-1)^3] = \cdots = k(k-1)(k-2),$$
$$f_4(k) = (k-1)[(k-1)^3 + (-1)^4] = \cdots = k(k-1)[(k-1) + (k-2)^2].$$

We first check that these formulas are correct. For $n = 3$, simply note that if we have only three vertices then they must all get different colours, so the number of possible colourings is $P(k, 3) = k(k-1)(k-2)$, v.s.v. For $n = 4$, first note that vertices 0 and 1 must get different colours, so there are $P(k, 2) = k(k-1)$ possible ways to colour these two. When we come to colour vertices 2 and 3, there are then two cases: (i) 2 gets the same colour as 0. In this case, 3 can be given any other colour, so there are $k-1$ ways to complete the colouring (ii) 2 gets a different colour to 0. Here there are $k-2$ possible colours for 2, since it must also get a different colour to 1. Then 3 must get a different colour from both 0 and 2, hence it also has $k-2$ possible colours. By MP, there are $(k-2)^2$ ways to complete the colouring in case (ii). Finally, AP+MP yield the correct formula for $f_4(k)$.

Having verified the base cases, we just use induction and compute that

$$\begin{aligned} f_n(k) &= (k-1)\cdot(k-1)[(k-1)^{n-3}+(-1)^{n-2}] + (k-2)\cdot(k-1)[(k-1)^{n-2}+(-1)^{n-1}] = \\ &= (k-1)^{n-2}[(k-1)+(k-2)(k-1)] + (-1)^n[(k-1)^2-(k-2)(k-1)] = \\ &= (k-1)^n + (-1)^n\cdot(k-1) = (k-1)[(k-1)^{n-1}+(-1)^n], \quad \text{v.s.v.} \end{aligned}$$

6. By induction on n. If $n = 1$ then there is one vertex and so $k = k\cdot(k-1)^0$ ways to colour it, so the formula is correct in this case. Now consider any tree $\mathcal{T}$ with $n \geq 2$ vertices. Let v be any leaf. Removing v leaves a tree $\mathcal{T}'$ on $n-1$ vertices. By the induction hypothesis, $\mathcal{T}'$ can be coloured in $k(k-1)^{n-2}$ ways. There will then be $k-1$ possibilites for v's colour when reinserted into the tree, since it will have only one neighbor and it must get a different colour from that vertex. By MP, the total number of ways to colour $\mathcal{T}$ is $[k(k-1)^{n-2}]\cdot(k-1) = k(k-1)^{n-1}$, v.s.v.

10. **(i)** We can prove this by induction on n. For $n = 1$ it states that $f_2 = f_3 - 1$, which is correct since $f_2 = 1$ and $f_3 = 2$. Suppose the identity holds for $n = m$ and consider $n = m+1$:

$$\begin{aligned} \sum_{k=1}^{m+1} f_{2k} = \left(\sum_{k=1}^{m} f_{2k}\right) + f_{2m+2} &\overset{\text{induction}}{=} (f_{2m+1}-1) + f_{2m+2} = \\ &= (f_{2m+1}+f_{2m+2}) - 1 = f_{2m+3} - 1, \quad \text{v.s.v.} \end{aligned}$$

(ii) I don't see any easier way to prove this than to use the explicit formula for f_n derived in class (see also Ex. 19.2.2 in Biggs),

$$f_n = \frac{1}{\sqrt{5}}\left(\gamma^n + (-1)^{n+1}\gamma^{-n}\right), \quad \gamma^{\pm 1} = \frac{\sqrt{5}\pm 1}{2}.$$

The RHS of our desired equation is

$$f_{3n} = \frac{1}{\sqrt{5}}\left(\gamma^{3n} + (-1)^{3n+1}\gamma^{-3n}\right).$$

The LHS can be manipulated as follows:

$$f_{n+1}^3 + f_n^3 - f_{n-1}^3 = \left[\frac{1}{\sqrt{5}}\left(\gamma^{n+1} + (-1)^{n+2}\gamma^{-(n+1)}\right)\right]^3 +$$
$$+\left[\frac{1}{\sqrt{5}}\left(\gamma^{n} + (-1)^{n+1}\gamma^{-n}\right)\right]^3 + \left[\frac{1}{\sqrt{5}}\left(\gamma^{n-1} + (-1)^{n}\gamma^{-(n-1)}\right)\right]^3 =$$
$$= \frac{1}{5\sqrt{5}}(A+B+C+D),$$

where

$$A = \gamma^{3n+3} + \gamma^{3n} - \gamma^{3n-3},$$
$$B = 3[(-1)^{n+2}\gamma^{n+1} + (-1)^{n+1}\gamma^{n} - (-1)^{n}\gamma^{n-1}],$$
$$C = 3[\gamma^{-(n+1)} + \gamma^{-n} - \gamma^{-(n-1)}],$$
$$D = (-1)^{3n+6}\gamma^{-(3n+3)} + (-1)^{3n+3}\gamma^{-3n} - (-1)^{3n}\gamma^{-(3n-3)}.$$

It now suffices to show that

$$A = 5\gamma^{3n}, \quad B = C = 0, \quad D = 5\cdot(-1)^{3n+3}\gamma^{-3n}.$$

Let's start with A. We can write it as $A = \gamma^{3n}(\gamma^3 + 1 - \gamma^{-3})$. On the one hand,

$$\gamma^3 = \gamma\cdot\gamma^2 = \gamma(\gamma+1) = \gamma^2 + \gamma = (\gamma+1) + \gamma = 2\gamma + 1.$$

On the other hand,

$$\gamma^{-1} = \gamma - 1 \Rightarrow \gamma^{-2} = 1 - \gamma^{-1} = 1 - (1-\gamma) = 2 - \gamma \Rightarrow$$
$$\Rightarrow \gamma^{-3} = (\gamma-1)(2-\gamma) = -\gamma^2 + 3\gamma - 2 = -(\gamma+1) + 3\gamma - 2 = 2\gamma - 3.$$

Thus $\gamma^3 + 1 - \gamma^{-3} = (2\gamma+1) + 1 - (2\gamma - 3) = 5$, which proves that $A = 5\cdot\gamma^{3n}$.

Next consider B. We can write it as

$$B = 3\cdot(-1)^n\gamma^{n-1}[-1 - \gamma + \gamma^2] = 0, \quad \text{since } \gamma^2 = \gamma + 1, \text{ v.s.v.}$$

Similarly,

$$C = 3\gamma^{-(n+1)}[1 + \gamma - \gamma^2] = 0.$$

Finally,

$$D = (-1)^{3n+3}\gamma^{-3n}[(-1)^3\gamma^{-3} + 1 - (-1)^{-3}\gamma^3] =$$
$$= (-1)^{3n+3}\gamma^{-3n}[-\gamma^{-3} + 1 + \gamma^3] \overset{\text{see } A}{=} 5\cdot(-1)^{3n+3}\gamma^{-3n}, \quad \text{v.s.v.}$$

11. CASE 1: A subset contains n. Then it can't contain $n-1$, so must contain $k-1$ of the numbers up to $n-2$. Thus $\lambda(n-2,\, k-1)$ possibilities.

CASE 2: A subset doesn't contain n. Then it contains k of the numbers up to $n-1$, so $\lambda(n-1,\, k)$ possibilities.

Then the addition principle yields the recursion. The explicit formula can be verified by induction on $n+k$. Since $\lambda(n-2,\, k-1)$ appears in the recursion our base case is all pairs $(n,\, k)$ such that $n+k \le 3$. These pairs are $(1,\, 0)$, $(1,\, 1)$, $(2,\, 0)$, $(2,\, 1)$, $(3,\, 0)$. It is easy to see directly that

- $\lambda(n,\, 0) = 1$ since the only possible subset is the empty set, for any n. This accords with the formula: $\lambda(n,\, 0) = \binom{n-0+1}{0} = \binom{n+1}{0} = 1$.

- $\lambda(n,\,1) = n$, since any of the n elements can be chosen. This accords with the formula: $\lambda(n,\,1) = \binom{n-1+1}{1} = \binom{n}{1} = n$.

The induction step is then carried out as follows:

$$\begin{aligned}\lambda(n,\,k) &= \lambda(n-2,\,k-1) + \lambda(n-1,\,k) = \\ &= \binom{(n-2)-(k-1)+1}{k-1} + \binom{(n-1)-k+1}{k} = \\ &= \binom{n-k}{k-1} + \binom{n-k}{k} \overset{(2.2)}{=} \binom{n-k+1}{k}, \quad \text{v.s.v.}\end{aligned}$$

To prove the formula directly, consider a k-subset of $\{1,\,\dots,\,n\}$ satisfying our requirements. List its elements in increasing order as $1 \leq x_1 < x_2 < \cdots < x_k \leq n$. Set

$$y_1 = x_1 - 1; \quad y_i = x_i - x_{i-1} - 2, \;\; i = 2,\,\dots,\,k; \quad y_{k+1} = n - x_k.$$

Then

$$\sum_{i=1}^{k+1} y_i = n - (2k-1) \quad \text{and} \quad y_i \geq 0 \;\; \forall\, i, \tag{0.1}$$

where the latter is due to the fact that no two x_i are consecutive. Moreover there is a $1-1$ correspondence between the sequences $(x_1,\,\dots,\,x_k)$ satisfying our requirements and the sequences $(y_1,\,\dots,\,y_{k+1})$ satisfying (0.1). Thus $\lambda(n,\,k)$ equals the number of latter sequences which, by Example 2.11 in the lecture notes, is $\binom{(n-(2k-1))+(k+1)-1}{(k+1)-1} = \binom{n-k+1}{k}$, v.s.v.

12. Label the numbers clockwise as $1,\,2,\,\dots,\,n$ and consider two cases:

CASE 1: 1 is chosen. Then neither n nor 2 can be chosen and we still have to choose $k-1$ of the $n-3$ numbers $3,\,4,\,\dots,\,n-1$, in such a way that we never choose two which are adjacent. But now "adjacent" is the same thing on the real line as on the circle, so there are $\lambda(n-3,\,k-1)$ possibilities.

CASE 2: We don't choose 1. Then k of the $n-1$ numbers $2,\,3,\,\dots,\,n$ must be chosen in such a way that no two which are adjacent are chosen. Once again, "adjacent" now means the same thing on the real line as on the circle so there are $\lambda(n-1,\,k)$ possibilities.

The addition principle yields the desired recursion for $\mu(n,\,k)$. Then using the formula for $\lambda(\cdot,\,\cdot)$ from Ex. 11 and noting that

$$\binom{n-k-1}{k-1} = \frac{(n-k-1)!}{(k-1)!(n-2k)!} = \frac{(n-k)!}{k!(n-2k)!} \times \frac{k}{n-k} = \binom{n-k}{k} \times \frac{k}{n-k}, \tag{0.2}$$

we compute as follows:

$$\begin{aligned}\mu(n,\,k) = \lambda(n-1,\,k) + \lambda(n-3,\,k-1) &= \binom{(n-1)-k+1}{k} + \binom{(n-3)-(k-1)+1}{k-1} = \\ &= \binom{n-k}{k} + \binom{n-k-1}{k-1} \overset{(0.2)}{=} \binom{n-k}{k}\left[1 + \frac{k}{n-k}\right] = \frac{n}{n-k}\binom{n-k}{k}, \quad \text{v.s.v.}\end{aligned}$$