EXERCISES: CHAPTER 25

Section 25.3 (The binomial theorem for negative exponents)

1. Write down, and simplify wherever possible, the coefficient of

(i) x^3 in $(1+2x)^{-7}$
(ii) x^n in $(1-x)^{-4}$
(iii) x^{2r} in $(1-x)^{-r}$.

2. Write down the first four terms and the general term in the power series $(1-x)^{-3}$.

3. Let a_n be the coeffiocient of x^n in the power series $(1-x-x^2)^{-1}$. Show that

$$a_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-r}{r},$$

where r is the greatest integer such that $0 \le r \le n-r$.

4. What is the coefficient of x^n in the power series

$$\frac{1+2x+2x^2}{1-3x+3x^2-x^3} \quad ?$$

5. If $a_0,\ a_1,\ \ldots,\ a_{6r}$ and $b_0,\ b_1,\ \ldots,\ b_{3r}$ are defined by

$$(1-x+x^2)^{3r} = \sum_{i=0}^{6r} a_i x^i,$$

$$(1+x)^{3r} = \sum_{i=0}^{3r} b_i x^i,$$

show that

$$\sum_{i=0}^{3r} a_i b_{3r-i} = \binom{3r}{r}.$$

Section 25.4 (Generating functions)

1. Use the generating function method to find a formula for u_n when the sequence (u_n) is defined by

$$u_o = 1, \quad u_1 = 1, \quad u_{n+2} - 4u_{n+1} + 4u_n = 0 \ \ \forall\, n \ge 0.$$

2. Suppose $A(x)$ is the generating function for the sequence (a_n). What are the generating functions for the sequences (p_n), (q_n) and (r_n) defined as follows ?:

$$\text{(i) } p_n = 5a_n; \quad \text{(ii) } q_n = a_n + 5; \quad \text{(iii) } r_n = a_{n+5}.$$

3. Show that $x(1+x)/(1-x)^3$ is the generating function for the sequence whose nth term is n^2.

4. Let $A(x)$ be the generating function for the sequence (a_n) and define $s_n = \sum_{i=0}^{n} a_i$. Show that the generating function for (s_n) is $S(x) = \frac{A(x)}{1-x}$. Use this result in conjunction with Ex. 3 to find a formula for $\sum_{i=0}^{n} i^2$.

Section 25.5 (The homogeneous linear recursion)

1. Use the auxiliary equation method to find a formula for u_n when the sequence (u_n) is defined by

(i) $u_0 = 1$, $u_1 = 3$, $u_{n+2} - 3u_{n+1} - 4u_n = 0$ $(n \geq 0)$
(ii) $u_0 = 2$, $u_1 = 0$, $u_2 = -2$, $u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 0$ $(n \geq 0)$
(iii) $u_0 = 1$, $u_1 = 0$, $u_2 = 0$, $u_{n+3} - 3u_{n+1} + 2u_n = 0$ $(n \geq 0)$.

2. Professor McBrain climbs stairs in an erratic fashion. Sometimes he takes two stairs in one stride, sometimes only one. Find a formula for b_n, the number of different ways in which he can climb n stairs.

3. Suppose the sequence (z_n) is defined by

$$z_0 = 1, \quad z_{n+1} = \frac{z_n - a}{z_n - b} \ \ \forall\, n \geq 0,$$

where a and b are real numbers and $b \neq 1$. Show that if the sequence (u_n) satisfies $\frac{u_{n+1}}{u_n} = z_n - b$ then

$$u_{n+2} + (b-1)u_{n+1} + (a-b)u_n = 0 \ \ \forall\, n \geq 0.$$

Hence find a formula for z_n when $a = 0$ and $b = 2$.

4. Let (u_n), (v_n) and (w_n) be the sequences defined by $u_0 = v_0 = w_0 = 1$ and

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} 4 & -3 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} \quad \forall\, n \geq 0.$$

Show that (u_n) satisfies a homogeneous linear recursion and hence find a formula for u_n.

Section 25.6 (Non-homogeneous linear recursions)

1. Show that the generating function for the sequence (u_n) defined by the recursion

$$u_0 = 1, \quad u_{n+1} - 2u_n = 4^n \;\; \forall\, n \geq 0$$

is

$$U(x) = \frac{1-3x}{(1-2x)(1-4x)}.$$

Deduce that $u_n = 2^{2n-1} + 2^{n-1}$.

2. Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b's. Show that

$$q_{n+1} = 4^n + 2q_n \;\; \forall\, n \geq 1.$$

Hence find the generating function $Q(x)$ for (q_n), assuming $q_0 = 0$, and show that $q_n = \frac{1}{2}(4^n - 2^n)$.

3. Show that the generating function for the sequence defined by

$$u_0 = 1, \quad u_{n+1} - 2u_n = n\alpha^n \;\; \forall\, n \geq 0$$

is

$$u(x) = \frac{1}{1-2x} + \frac{\alpha x^2}{(1-2x)(1-\alpha x)^2},$$

provided $\alpha \neq 2$. Hence find a formula for u_n. What happens when $\alpha = 2$?

Review Section 25.7

2. Write down the first four terms and the general term in the power series for $(1+x)^{-5}$.

3. Find the coefficient of x^n in the power series for

$$\frac{26 - 60x + 25x^2}{(1-2x)(1-5x)^2}.$$

4. Find the first four terms and the coefficient of x^n in the power series for

$$\frac{1 - x - x^2}{(1-2x)(1-x)^2}.$$

7. Use the method of Ex. 25.4.4 to find formulae for $\sum_{i=1}^{n} i^3$ and $\sum_{i=1}^{n} i^4$.

9. By taking the derivative of the binomial expansion show that $\sum_{k=1}^{n} k\binom{n}{k} = 2^{n-1}n$.

10. Use the formula $(1-x^2)^{-n} = (1-x)^{-n}(1+x)^{-n}$ to prove that

$$\sum_{i=0}^{r} (-1)^i \binom{n+r-i-1}{r-i}\binom{n+i-1}{i} = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ \binom{n+r/2-1}{r/2}, & \text{if } r \text{ is even.} \end{cases}$$

11. Find an expression for $(1-x)^{-n}(1-x^k)^n$ as a polynomial of degree $n(k-1)$ and hence prove that, when $r \geq nk$,

$$\sum_{i=0}^{n}(-1)^i\binom{n+r-ik-1}{n-1}\binom{n}{i} = 0.$$

12. Write down the generating function for the sequence (u_n) whose terms represent the number of ways of distributing n different books to four people.

13. Let c_r be the number of ways in which the total r can be obtained when four dice are thrown. Explain why the generating function for the sequence (c_r) is

$$C(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

14. Write down the generating function for the numbers b_r of integers n in the range $0 \leq n \leq 10^m - 1$ for which the sum of the base-10 digits is r.

15. Use the generating function method to solve the following recursion:

$$u_0 = 2, \quad u_1 = -6, \quad u_{n+2} + 8u_{n+1} - 9u_n = 8(3^{n+1}) \ \ \forall\, n \geq 0.$$

16. Find the general form of the solution to the recursion

$$y_{n+2} - 6y_{n+1} + 9y_n = 2^n + n \ \ \forall\, n \geq 0.$$

17. Let $\lambda(n,\, k)$ be the numbers in Ex. 19.7.11. For fixed k let $F_k(x)$ be the generating function of the sequence $(\lambda(n,\, k))_{n=0}^{\infty}$. Find a recursion for $F_k(x)$ and use this to give another proof that $\lambda(n,\, k) = \binom{n-k+1}{k}$.

18. Use the generating function method to solve the recursion

$$u_0 = 1, \quad u_{n+1} = 3u_n + 2^{n-1} \ \ \forall\, n \geq 0.$$

19. The *exponential generating function (EGF)* for the sequence (u_n) is defined to be the power series

$$U(x) = \sum_{n=0}^{\infty} \frac{u_n}{n!}\, x^n.$$

Use the recursion established in Ex. 19.1.3 to show that the EGF for the sequence (d_n) is $e^{-x}/(1-x)$. Use this to give another proof of the formula (3.10).

20. Let $\hat{Q}(x)$ be the EGF for the numbers q_n, where q_n is the total number of partitions of an n-set. Use the formula obtained in Ex. 12.7.10 to prove that $\hat{Q}(x) = \exp(e^x - 1)$.