ANSWERS: CHAPTER 25

Section 25.3 (The binomial theorem for negative exponents)

1. (i) $2^3 \times (-1)^3 \binom{7+3-1}{3} = -8 \times \binom{9}{3} = -672$.
(ii) $\binom{4+n-1}{n} = \binom{n+3}{3} = \frac{(n+1)(n+2)(n+3)}{6}$.
(iii) $\binom{r+2r-1}{2r} = \binom{3r-1}{r-1}$.

2.

$$(1-x)^{-3} = \sum_{n=0}^{\infty} \binom{3+n-1}{n} x^n = \sum_{n=0}^{\infty} \binom{n+2}{n} x^n =$$

$$= \sum_{n=0}^{\infty} \binom{n+2}{2} x^n = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n = 1 + 3x + 6x^2 + 10x^3 + \dots.$$

3. First note that

$$(1-x-x^2)^{-1} = \sum_{m=0}^{\infty} (x+x^2)^m = \sum_{m=0}^{\infty} [x(1+x)]^m =$$
$$= \sum_{m=0}^{\infty} x^m (1+x)^m = \sum_{m=0}^{\infty} x^m \left(\sum_{k=0}^{m} \binom{m}{k} x^k \right).$$

For each pair (m, k) such that $0 \le k \le m$ and $m + k = n$ we will get a contribution of $\binom{m}{k}$ to the coefficient of x^n. Note that $m + k = n \Rightarrow k = n - m$ so $k \le m$ iff $n - m \le m$, i.e.: iff $m \ge n/2$. Hence the total coefficient of x^n will be

$$\sum_{n/2 \le m \le n} \binom{m}{n-m} \overset{m:=n-l}{=} \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-l}{l}, \quad \text{v.s.v.}$$

4. The denominator is $(1-x)^3$ so the rational function expands (see Ex. 2) as

$$(1+2x+2x^2)(1-x)^{-3} = (1+2x+2x^2) \left(\sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{2} x^m \right).$$

Hence the coefficient of x^n will be

$$\frac{(n+1)(n+2)}{2} + 2 \times \frac{n(n+1)}{2} + 2 \times \frac{(n-1)n}{2} = \cdots = \frac{5n^2+3n+2}{2}.$$

5. The LHS is, by definition of polynomial multiplication, the coefficient of x^{3r} in $(\sum a_i x^i)(\sum b_i x^i)$. But

$$\left(\sum_{i=0}^{6r} a_i x^i\right)\left(\sum_{i=0}^{3r} b_i x^i\right) = (1-x+x^2)^{3r}(1+x)^{3r} =$$

$$= [(1-x+x^2)(1+x)]^{3r} = (1+x^3)^{3r} = \sum_{k=0}^{3r}\binom{3r}{k}x^{3k}.$$

Hence the coefficient of x^{3r} is $\binom{3r}{r}$, v.s.v.

Section 25.4 (Generating functions)

1. Let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 1$ we can write, firstly,

$$U(x) = 1 + \sum_{n=1}^{\infty} u_n x^n = 1 + x\left(\sum_{n=0}^{\infty} u_{n+1}\, x^n\right) \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} u_{n+1}\, x^n = \frac{U(x)-1}{x}. \tag{0.1}$$

Then, using also $u_1 = 1$ and the recursion,

$$U(x) = 1 + x + \sum_{n=2}^{\infty} u_n x^n = (1+x) + x^2\left(\sum_{n=0}^{\infty} u_{n+2}\, x^n\right) =$$

$$= (1+x) + x^2\left(4\sum_{n=0}^{\infty} u_{n+1}x^n - 4\sum_{n=0}^{\infty} u_n x^n\right) =$$

$$\overset{(0.1)}{=} (1+x) + x^2\left(4 \times \frac{U(x)-1}{x} - 4U(x)\right) \Rightarrow$$

$$\Rightarrow U(x) = (1+x) + 4x(U(x)-1) - 4x^2U(x) \Rightarrow$$

$$\Rightarrow (1-4x+4x^2)U(x) = 1-3x \Rightarrow U(x) = \frac{1-3x}{1-4x+4x^2}.$$

But then, using the Binomial Theorem,

$$\frac{1-3x}{1-4x+4x^2} = \frac{1-3x}{(1-2x)^2} = (1-3x)(1-2x)^{-2} =$$

$$= (1-3x)\left(\sum_{m=0}^{\infty}\binom{m+2-1}{m}(2x)^m\right) = (1-3x)\left(\sum_{m=0}^{\infty}(m+1)\cdot 2^m \cdot x^m\right).$$

Hence the coefficient of x^n, which by definition is equal to u_n, is given by

$$u_n = (n+1)2^n - 3n2^{n-1} = \left(1 - \frac{n}{2}\right)2^n.$$

2. Call the asked for generating functions $P(x)$, $Q(x)$, $R(x)$ respectively. By definition, $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i)

$$P(x)=\sum_{n=0}^{\infty} p_n x^n=\sum_{n=0}^{\infty}(5a_n)x^n=5\sum_{n=0}^{\infty} a_n x^n=5A(x).$$

(ii)

$$Q(x)=\sum_{n=0}^{\infty} q_n x^n=\sum_{n=0}^{\infty}(a_n+5)x^n=\sum_{n=0}^{\infty} a_n x^n+5\sum_{n=0}^{\infty} x^n=A(x)+\frac{5}{1-x}.$$

(iii) First note that

$$\sum_{n=5}^{\infty} a_n x^n=x^5\left(\sum_{n=5}^{\infty} a_n x^{n-5}\right)=x^5\left(\sum_{n=0}^{\infty} a_{n+5}\, x^n\right)=x^5 R(x).$$

Thus,

$$A(x)=(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4)+x^5R(x)\Rightarrow$$
$$\Rightarrow R(x)=\frac{A(x)-(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4)}{x^5}.$$

3. I will give two different solutions to this problem for the sake of pedagogy.

First Solution: Use the binomial theorem (see Ex. 25.3.2):

$$\frac{x(1+x)}{(1-x)^3}=x(1+x)(1-x)^{-3}=(x+x^2)\left(\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2}x^m\right).$$

The coefficient of x^n has contributions from $m=n-1$ and $m=n-2$ and is thus $\frac{n(n+1)}{2}+\frac{(n-1)n}{2}=n^2$, v.s.v.

Second Solution: Let $G(x):=\sum_{n=0}^{\infty} n^2x^n$. Firstly,

$$G(x)=0+x+x^2\sum_{n=2}^{\infty} n^2x^{n-2}\Rightarrow\sum_{n=2}^{\infty} n^2x^{n-2}=\frac{G(x)}{x^2}-\frac{1}{x}. \tag{0.2}$$

Secondly, by the Binomial Theorem,

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n.$$

Differentiating both sides once gives

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty} n\,x^{n-1}=1+x\left(\sum_{n=2}^{\infty} n\,x^{n-2}\right)\Rightarrow$$
$$\Rightarrow\sum_{n=2}^{\infty} n\,x^{n-2}=\frac{1}{x(1-x)^2}-\frac{1}{x}. \tag{0.3}$$

Differentating a second time gives

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n^2 x^{n-2} - \sum_{n=2}^{\infty} n\, x^{n-2} =$$

$$\overset{(0.2)\,+\,(0.3)}{=} \left(\frac{G(x)}{x^2} - \frac{1}{x}\right) - \left(\frac{1}{x(1-x)^2} - \frac{1}{x}\right) \Rightarrow$$

$$\Rightarrow G(x) = x^2 \left(\frac{2}{(1-x)^3} + \frac{1}{x(1-x)^2}\right) = \cdots = \frac{x(1+x)}{(1-x)^3} \text{ v.s.v.}$$

4.

$$\frac{A(x)}{1-x} = A(x) \cdot (1-x)^{-1} = \left(\sum_{l=0}^{\infty} a_l x^l\right) \left(\sum_{m=0}^{\infty} x^m\right).$$

When we multiply out, we'll get a contribution of a_l to the coefficient of x^n from $(a_l x^l) \cdot x^{n-l}$. Thus, we'll get such a contribution for every $l = 0, 1, \ldots, n$ which means that the total coefficient of x^n is $\sum_{l=0}^{n} a_l = s_n$, v.s.v.

From Ex. 3 we know that

$$A(x) = \sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

Let $s_n := \sum_{i=0}^{n} i^2$ and $S(x) := \sum_{n=0}^{\infty} s_n x^n$. By Ex. 3 we have

$$S(x) = \frac{A(x)}{1-x} = \frac{x(1+x)}{(1-x)^4} =$$

$$= (x+x^2)\left(\sum_{m=0}^{\infty} \binom{m+4-1}{m} x^m\right) = (x+x^2)\left(\sum_{m=0}^{\infty} \frac{(m+1)(m+2)(m+3)}{6} x^m\right).$$

There are contributions to the coefficient of x^n coming from $m = n-1$ and $m = n-2$. Thus the total coefficient is

$$\frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)}{6} = \frac{n(n+1)}{6}\left((n+2)+(n-1)\right) = \frac{n(n+1)(2n+1)}{6}.$$

In other words, we recover the well-known formula

$$s_n = \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Section 25.5 (The homogeneous linear recursion)

1. (i) The auxiliary equation is $x^2 - 3x - 4 = 0$, which has roots $x_1 = 4$, $x_2 = -1$. Hence the general solution is

$$u_n = C_1 \cdot 4^n + C_2 \cdot (-1)^n.$$

Inserting the initial conditions yields the two equations

$$C_1 + C_2 = 1, \quad 4C_1 - C_2 = 3,$$

whose solution is $C_1 = 4/5$, $C_2 = 1/5$. Hence,

$$u_n = \frac{1}{5}\left(4^{n+1} + (-1)^n\right).$$

(ii) The auxiliary equation is $x^3 - 6x^2 + 11x - 6 = 0$, which has roots $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. Hence the general solution is

$$u_n = C_1 + C_2 \cdot 2^n + C_3 \cdot 3^n.$$

Inserting the initial conditions yields the three equations

$$C_1 + C_2 + C_3 = 2, \quad C_1 + 2C_2 + 3C_3 = 0, \quad C_1 + 4C_2 + 9C_3 = -2,$$

whose solution (after some boring Gauss elimination !) is $C_1 = 5$, $C_2 = -4$, $C_3 = 1$. Hence,

$$u_n = 5 - 2^{n+2} + 3^n.$$

(iii) The auxiliary equation is $x^3 - 3x + 2 = 0$, which has roots $x_1 = x_2 = 1$, $x_3 = -2$. Hence the general solution is

$$u_n = C_1 + C_2 \cdot n + C_3 \cdot (-2)^n.$$

Inserting the initial conditions yields the three equations

$$C_1 + C_3 = 1, \quad C_1 + C_2 - 2C_3 = 0, \quad C_1 + 2C_2 + 4C_3 = 0,$$

whose solution is $C_1 = 8/9$, $C_2 = -2/3$, $C_3 = 1/9$. Hence,

$$u_n = \frac{1}{9}\left(8 - 6n + (-2)^n\right).$$

2. It's easy to check that $b_1 = 1$, $b_2 = 2$ and $b_{n+2} = b_{n+1} + b_n$. Hence $b_n = f_{n+1} = \frac{1}{\sqrt{5}}\left(\gamma^{n+1} + (-1)^{n+2}\gamma^{-(n+1)}\right)$.

3. We have $u_{n+1} = (z_n - b)u_n$ and

$$u_{n+2} = (z_{n+1} - b)u_{n+1} = \left(\frac{z_n - a}{z_n - b} - b\right)(z_n - b)u_n = ((z_n - a) - b(z_n - b))u_n.$$

Hence,

$$u_{n+2} + (b-1)u_{n+1} + (a-b)u_n =$$
$$= u_n\left[(z_n - a) - b(z_n - b) + (b-1)(z_n - b) + (a - b)\right] = \cdots = u_n[0] = 0, \quad \text{v.s.v.}$$

When $a = 0$ and $b = 2$, the recurrence for u_n becomes $u_{n+2} + u_{n+1} - 2u_n = 0$. This has auxiliary equation $x^2 + x - 2 = 0$ and roots $x_1 = 1$, $x_2 = -2$, so $u_n = C_1 + C_2 \cdot (-2)^n$. Turning to (z_n) we have

$$z_n = \frac{u_{n+1}}{u_n} + b = \frac{C_1 + C_2 \cdot (-2)^{n+1}}{C_1 + C_2 \cdot (-2)^n} + 2. \tag{0.4}$$

We have the intitial condition $z_0 = 1$, whose insertion gives

$$\frac{C_1 - 2C_2}{C_1 + C_2} + 2 = 1 \Rightarrow \cdots \Rightarrow C_2 = 2C_1.$$

Plugging this back into (0.4), we get

$$z_n = \frac{C_1(1 + 2 \cdot (-2)^{n+1})}{C_1(1 + 2 \cdot (-2)^n)} + 2 = \frac{1 + 2 \cdot (-2)^{n+1}}{1 + 2 \cdot (-2)^n} + 2 = \cdots = \frac{3}{1 + 2 \cdot (-2)^n}.$$

4. Denote $\boldsymbol{x}_n = \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix}$ and $A = \begin{bmatrix} 4 & -3 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. Then the recursion can be written as

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_{n+1} = A\boldsymbol{x}_n \ \ \forall\, n \geq 0.$$

The solution is $\boldsymbol{x}_n = A^n \boldsymbol{x}_0$. From linear algebra we know that, if A has distinct eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ then it is diagonalisable, and if $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3$ are the corresponding eigenvectors then the solution can be written explicitly as

$$\boldsymbol{x}_n = C_1 \cdot \lambda_1^n \cdot \boldsymbol{v}_1 + C_2 \cdot \lambda_2^n \cdot \boldsymbol{v}_2 + C_3 \cdot \lambda_3^n \cdot \boldsymbol{v}_3, \quad \text{where} \quad \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]^{-1} \boldsymbol{x}_0. \tag{0.5}$$

To find the eigenvalues and eigenvectors of A is a standard linear algebra exercise, so let me just give you the answer:

$$\lambda_1 = 1,\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = 2,\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad \lambda_3 = 3,\ \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, in turn,

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \cdots = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Substituting everything into (0.5) we obtain

$$\begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} = \begin{bmatrix} 1 + 2^{n+1} - 2 \cdot 3^n \\ 1 \\ 2^{n+1} - 3^n \end{bmatrix}.$$

Section 25.6 (Non-homogeneous linear recursions)

1. Let $U(x) = \sum_{n=0}^{\infty} u_n x^n$. From the recursion we obtain

$$U(x) = 1 + x \left(\sum_{n=0}^{\infty} u_{n+1}\, x^n \right) = 1 + x \left(\sum_{n=0}^{\infty} (2u_n + 4^n) x^n \right) =$$
$$= 1 + x \left(2 \cdot U(x) + \sum_{n=0}^{\infty} (4x)^n \right) = 1 + x \left(2 \cdot U(x) + \frac{1}{1-4x} \right).$$

Thus,

$$(1-2x)U(x) = 1 + \frac{x}{1-4x} = \frac{1-3x}{1-4x} \Rightarrow U(x) = \frac{1-3x}{(1-2x)(1-4x)}, \quad \text{v.s.v.}$$

We next seek a partial fraction decomposition

$$\begin{aligned}\frac{1-3x}{(1-2x)(1-4x)} &= \frac{A}{1-2x}+\frac{B}{1-4x}\\ \Leftrightarrow 1-3x = A(1-4x)+B(1-2x) &= (A+B)+x(-4A-2B)\\ \Leftrightarrow A+B=1 \text{ and } 4A+2B=3 \Rightarrow A &= B = \frac{1}{2}.\end{aligned}$$

Hence, using the binomial theorem,

$$U(x)=\frac{1}{2}\left(\frac{1}{1-2x}+\frac{1}{1-4x}\right)=\frac{1}{2}\left(\sum_{n=0}^{\infty}(2x)^n+\sum_{n=0}^{\infty}(4x)^n\right).$$

The coefficient of x^n, namely u_n, is thus $\frac{1}{2}(2^n+4^n)=2^{n-1}+2^{2n-1}$, v.s.v.

2. To verify the recursion, we consider an admissable word of length $n+1$ and consider two cases:

CASE 1: The first letter is b. Then the remainder of the word, consisting of n letters, must contain an even number of b's. There are a total of 4^n words of length n in the alphabet and q_n of these have an odd number of b's. Hence 4^n-q_n of them have an even number of b's.

CASE 2: The first letter is not b. Then there are 3 choices for the first letter. The remaining n letters constitute a word with an odd number of b's, so there are q_n possibilities for it. By MP, there are a total of $3q_n$ possible words in this case.

By AP, it follows that $q_{n+1}=(4^n-q_n)+3q_n=2q_n+4^n$, v.s.v. This is the same recursion as in Ex. 1, except that the initial condition $q_0=0$ is slightly different. However, the computations will be very similar to Ex. 1, so let me just note the main points. First, the generating function $Q(x)$ will turn out to be

$$Q(x)=\frac{x}{(1-2x)(1-4x)}.$$

The partial fraction decomposition will be

$$\frac{x}{(1-2x)(1-4x)}=\frac{1}{2}\left(\frac{-1}{1-2x}+\frac{1}{1-4x}\right),$$

and hence $Q(x)=\frac{1}{2}[\sum_{n=0}^{\infty}(4^n-2^n)\,x^n]$, v.s.v.

3. First assume $\alpha\neq 2$. Let $u(x)=\sum_{n=0}^{\infty}u_nx^n$. From the recursion we obtain

$$\begin{aligned}u(x)=1+x\left(\sum_{n=0}^{\infty}u_{n+1}\,x^n\right) &= 1+x\left(\sum_{n=0}^{\infty}(2u_n+n\alpha^n)x^n\right)=\\ &= 1+x\left(2\cdot u(x)+\sum_{n=0}^{\infty}n(\alpha x)^n\right).\end{aligned}\tag{0.6}$$

Now

$$\frac{1}{1-\alpha x}=\sum_{n=0}^{\infty}(\alpha x)^n.$$

Differentiating both sides we get

$$\frac{\alpha}{(1-\alpha x)^2} = \alpha\left(\sum_{n=1}^{\infty} n(\alpha x)^{n-1}\right) \Rightarrow$$
$$\cdots \Rightarrow \sum_{n=0}^{\infty} n(\alpha x)^n = \frac{\alpha x}{(1-\alpha x)^2}.$$

Substituting into (0.6) we get

$$u(x) = 1 + x\left(2\cdot u(x) + \frac{\alpha x}{(1-\alpha x)^2}\right),$$

which after a little algebra reduces to

$$u(x) = \frac{1}{1-2x} + \frac{\alpha x^2}{(1-2x)(1-\alpha x)^2}, \quad \text{v.s.v.}$$

The next step is to find a partial fraction decomposition

$$\frac{\alpha x^2}{(1-2x)(1-\alpha x)^2} = \frac{A}{1-2x} + \frac{B}{1-\alpha x} + \frac{C}{1-\alpha x)^2}$$
$$\Leftrightarrow \alpha x^2 = A(1-\alpha x)^2 + B(1-2x)(1-\alpha x) + C(1-2x)$$
$$\Leftrightarrow \alpha x^2 = (A+B+C) + x(-2\alpha A - (\alpha+2)B - 2C) + x^2(\alpha^2 A + 2\alpha B)$$
$$\Leftrightarrow A+B+C=0, \quad 2\alpha A + (\alpha+2)B + 2C = 0, \quad \alpha^2 A + 2\alpha B = \alpha.$$

After boring elimination we get

$$A = \frac{\alpha}{(\alpha-2)^2}, \quad B = \frac{-2(\alpha-1)}{(\alpha-2)^2}, \quad C = \frac{1}{\alpha-2}.$$

Hence,

$$u(x) = \left(1 + \frac{\alpha}{(\alpha-2)^2}\right)\frac{1}{1-2x} - \left(\frac{2(\alpha-1)}{(\alpha-2)^2}\right)\frac{1}{1-\alpha x} + \left(\frac{1}{\alpha-2}\right)\frac{1}{(1-\alpha x)^2}.$$

After further messing with the binomial theorem we obtain the formula

$$u_n = \left(1 + \frac{\alpha}{(\alpha-2)^2}\right)2^n - \frac{2(\alpha-1)}{(\alpha-2)^2}\,\alpha^n + \frac{1}{\alpha-2}\,(n+1)\alpha^n.$$

Finally, let's deal with $\alpha = 2$. The derivation of the rational function $u(x)$ will be exactly as before and so we'll get

$$u(x) = \frac{1}{1-2x} + \frac{2x^2}{(1-2x)^3}.$$

So this time we can apply the binomial theorem directly and write

$$u(x) = \sum_{n=0}^{\infty} 2^n x^n + 2x^2 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}\,2^n x^n.$$

Comparing coefficients of x^n we conclude that

$$u_n = 2^n + 2\cdot\frac{(n-1)n}{2}\cdot 2^{n-2} = \cdots = (n^2-n+4)2^{n-2}.$$

Review Section 25.7

2.

$$(1+x)^{-5} = \sum_{n=0}^{\infty}(-1)^n \binom{n+5-1}{n} x^n = \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)(n+2)(n+3)(n+4)}{24}\, x^n =$$

$$\cdots = 1 - 5x + 15x^2 - 35x^3 + \ldots$$

3. Call the power series $U(x)$. We begin with a partial fraction decomposition

$$U(x) = \frac{26-60x+25x^2}{(1-2x)(1-5x)^2} = \frac{A}{1-2x} + \frac{B}{1-5x} + \frac{C}{(1-5x)^2}$$

$$\Rightarrow 26-60x+25x^2 = A(1-5x)^2 + B(1-2x)(1-5x) + C(1-2x)$$

$$\Rightarrow 26-60x+25x^2 = (A+B+C) + x(-10A-7B-2C) + x^2(25A+10B)$$

$$\Rightarrow A+B+C = 26, \quad 10A+7B+2C = 60, \quad 25A+10B = 25.$$

Gauss elimination yields $A=1$, $B=0$, $C=25$. Now apply the binomial theorem:

$$U(x) = \frac{1}{1-2x} + \frac{25}{(1-5x)^2} =$$

$$= \sum_{n=0}^{\infty} 2^n x^n + 25\sum_{n=0}^{\infty}(n+1)5^n x^n =$$

$$= \sum_{n=0}^{\infty}(2^n + (n+1)5^{n+2})x^n.$$

4. Same approach as Ex. 3. Call the power series $U(x)$. We begin with a partial fraction decomposition

$$U(x) = \frac{1-x-x^2}{(1-2x)(1-x)^2} = \frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$\Rightarrow 1-x-x^2 = A(1-x)^2 + B(1-2x)(1-x) + C(1-2x)$$

$$\Rightarrow 1-x-x^2 = (A+B+C) + x(-2A-3B-2C) + x^2(A+2B)$$

$$\Rightarrow A+B+C = 1, \quad 2A+3B+2C = 1, \quad A+2B = -1.$$

Gauss elimination yields $A=1$, $B=-1$, $C=1$. Now apply the binomial theorem:

$$U(x) = \frac{1}{1-2x} - \frac{1}{1-x} + \frac{1}{(1-x)^2} =$$

$$= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(n+1)x^n =$$

$$= \sum_{n=0}^{\infty}(2^n + n)x^n = 1 + 3x + 6x^2 + 11x^3 + \ldots.$$

7. It doesn't really add any insight to formulate the idea in terms of generating functions, so let me give you the method in its usual (and most direct) formulation, in case you haven't seen it before. We already know that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{0.7}$$

Step 1: Observe that $(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$. Hence,

$$\sum_{i=1}^{n}[(i+1)^4 - i^4] = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1.$$

The LHS is a telescoping sum and equals $(n+1)^4 - 1$. The RHS can be simplified using (0.7). It follows that

$$(n+1)^4 - 1 = 4\sum_{i=1}^{n} i^3 + n(n+1)(2n+1) + 2n(n+1) + n$$

$$\Rightarrow \cdots \Rightarrow \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2. \tag{0.8}$$

Step 2: Observe that $(i+1)^5 - i^5 = 5i^4 + 10i^3 + 10i^2 + 5i + 1$. Hence,

$$\sum_{i=1}^{n}[(i+1)^5 - i^5] = 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1.$$

The LHS is a telescoping sum and equals $(n+1)^5 - 1$. The RHS can be simplified using (0.7) and (0.8). It follows that

$$(n+1)^5 - 1 = 5\sum_{i=1}^{n} i^4 + \frac{5n^2(n+1)^2}{2} + \frac{5n(n+1)(2n+1)}{3} + \frac{5n(n+1)}{2} + n$$

$$\Rightarrow \cdots \Rightarrow \sum_{i=1}^{n} i^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}.$$

9. The simple binomial theorem (Theorem 2.1 with $y = 1$) states that, if $n \in \mathbb{N}$, then

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Differentiating both sides with respect to x gives

$$n(x+1)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}.$$

Setting $x = 1$ yields $n\,2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$, v.s.v.

10. The binomial theorem implies that

$$(1-x^2)^{-n} = \sum_{m=0}^{\infty} \binom{n+m-1}{m} x^{2m},$$
$$(1-x)^{-n} = \sum_{m=0}^{\infty} \binom{n+m-1}{m} x^{m},$$
$$(1+x)^{-n} = \sum_{m=0}^{\infty} (-1)^m \binom{n+m-1}{m} x^{m}.$$

Now consider the equation $(1-x^2)^{-n} = (1-x)^{-n}(1+x)^{-n}$ in terms of the above power series. On the LHS, the coefficient of x^r is 0 if r is odd, and is $\binom{n+r/2-1}{r/2}$ if r is even. On the RHS, we're multiplying two power series, so the total coefficient of x^r will be

$$\sum_{m=0}^{r} (-1)^m \binom{n+m-1}{m} \binom{n+r-m-1}{r-m},$$

which is exactly what is claimed in the exercise.

11. $(1-x^k) = (1-x)(1+x+\cdots+x^{k-1})$. Hence,

$$(1-x)^{-n}(1-x^k)^n = (1-x)^{-n}[(1-x)(1+x+\cdots+x^{k-1})]^n = (1+x+\cdots+x^{k-1})^n,$$

which is a polynomial of degree $n(k-1)$, as claimed. On the other hand, the binomial theorem states that

$$(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i = \sum_{i=0}^{\infty} \binom{n+i-1}{n-1} x^i, \tag{0.9}$$
$$(1-x^k)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{ki}. \tag{0.10}$$

Since the product is a polynomial of degree $n(k-1)$, the coefficient of x^r must be zero for any $r > n(k-1)$. If $r \geq nk$ then every term on the right of (0.10) will contribute to the coefficient of x^r in the product of (0.10) and (0.9). Hence, for any such r, it follows from (0.9) and (0.10) that

$$0 = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+(r-ki)-1}{n-1}, \quad \text{v.s.v.}$$

12. Since the books are "different", there are 4^n ways to distribute n of them to four people, i.e.: $u_n = 4^n$. Hence,

$$U(x) = \sum_{n=0}^{\infty} u_n x^n = \sum_{n=0}^{\infty} 4^n x^n = \frac{1}{1-4x}.$$

13. If we multiply out $C(x)$, then we'll get a term x^r for any quadruple $(a,\, b,\, c,\, d)$ such that we choose x^a, x^b, x^c, x^d from the first, second, third and fourth factors respectively and $a+b+c+d=r$. Since each factor contains only powers x^t, where $1 \leq t \leq 6$, we can interpret each such quadruple $(a,\, b,\, c,\, d)$ as a possible result of throwing the four dice in order so that the total is r. In other words, coefficient of x^r = number of possible quadruples = number of ways to get a total of r when four dice are thrown.

14. $G(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)^m$.

15. Let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 2$ we can write, firstly,

$$U(x) = 2 + \sum_{n=1}^{\infty} u_n x^n = 2 + x\left(\sum_{n=0}^{\infty} u_{n+1}\, x^n\right) \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} u_{n+1}\, x^n = \frac{U(x)-2}{x}. \tag{0.11}$$

Then, using also $u_1 = -6$ and the recursion,

$$\begin{aligned}
U(x) &= 2 - 6x + \sum_{n=2}^{\infty} u_n x^n = (2-6x) + x^2\left(\sum_{n=0}^{\infty} u_{n+2}\, x^n\right) = \\
&= (2-6x) + x^2\left(-8\sum_{n=0}^{\infty} u_{n+1}x^n + 9\sum_{n=0}^{\infty} u_n x^n + 24\sum_{n=0}^{\infty} 3^n x^n\right) = \\
&\overset{(0.11)}{=} (2-6x) + x^2\left(-8 \times \frac{U(x)-2}{x} + 9U(x) + \frac{24}{1-3x}\right) \Rightarrow \\
&\Rightarrow U(x) = (2-6x) - 8x(U(x)-2) + 9x^2U(x) + \frac{24x^2}{1-3x} \Rightarrow \\
&\Rightarrow (1+8x-9x^2)U(x) = (2+10x) + \frac{24x^2}{1-3x} = (1+9x)(1-x)U(x) \Rightarrow \\
&\Rightarrow U(x) = \frac{2+10x}{(1+9x)(1-x)} + \frac{24x^2}{(1-3x)(1+9x)(1-x)} =
\end{aligned}$$

$$\cdots = \frac{2+4x-6x^2}{(1-3x)(1+9x)(1-x)} = \frac{2(1+3x)(1-x)}{(1-3x)(1+9x)(1-x)} = \frac{2(1+3x)}{(1-3x)(1+9x)}.$$

Next, we make a partial fraction decomposition,

$$\begin{aligned}
\frac{2(1+3x)}{(1-3x)(1+9x)} &= \frac{A}{1-3x} + \frac{B}{1+9x} \\
\Rightarrow 2(1+3x) &= A(1+9x) + B(1-3x) \\
&\Rightarrow A + B = 2, \quad 9A - 3B = 6,
\end{aligned}$$

which yields $A = B = 1$. Hence,

$$U(x) = \frac{1}{1-3x} + \frac{1}{1+9x}.$$

Finally, using the Binomial Theorem,

$$\begin{aligned}
U(x) &= \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (-1)^n \cdot 9^n \cdot x^n \\
&\Rightarrow u_n = 3^n(1 + (-1)^n 3^n).
\end{aligned}$$

16. Let $Y(x) := \sum_{n=0}^{\infty} y_n x^n$. We don't know the value of y_0 but can in any case write, firstly,

$$Y(x) = y_0 + \sum_{n=1}^{\infty} y_n x^n = y_0 + x\left(\sum_{n=0}^{\infty} y_{n+1}\, x^n\right) \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} y_{n+1}\, x^n = \frac{Y(x) - y_0}{x}. \tag{0.12}$$

Then, using also y_1, the recursion and the fact that

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=0}^{\infty} (n+1) x^n - \sum_{n=0}^{\infty} x^n =$$
$$= \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}, \tag{0.13}$$

we can compute as follows:

$$Y(x) = y_0 + y_1 x + \sum_{n=2}^{\infty} y_n x^n = (y_0 + y_1 x) + x^2 \left(\sum_{n=0}^{\infty} y_{n+2}\, x^n\right) =$$
$$= (y_0 + y_1 x) + x^2 \left(6 \sum_{n=0}^{\infty} y_{n+1} x^n - 9 \sum_{n=0}^{\infty} y_n x^n + \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} n x^n\right) =$$
$$\overset{(0.12)+(0.13)}{=} (y_0 + y_1 x) + x^2 \left(6 \times \frac{Y(x) - y_0}{x} - 9Y(x) + \frac{1}{1-2x} + \frac{x}{(1-x)^2}\right) \Rightarrow$$
$$\Rightarrow Y(x) = (y_0 + y_1 x) + 6x(Y(x) - y_0) - 9x^2 Y(x) + x^2 \left[\frac{1}{1-2x} + \frac{x}{(1-x)^2}\right] \Rightarrow$$
$$\Rightarrow (1 - 6x + 9x^2) Y(x) = (y_0 + (y_1 - 6y_0)x) + x^2 \left[\frac{1}{(1-2x)} + \frac{x}{(1-x)^2}\right] = (1-3x)^2 Y(x),$$
$$\Rightarrow Y(x) = \frac{y_0 + (y_1 - 6y_0)x}{(1-3x)^2} + x^2 \left[\frac{1}{(1-2x)(1-3x)^2} + \frac{x}{(1-x)^2(1-3x)^2}\right].$$

The second term in square brackets on the right will a priori have a partial fraction decomposition of the form

$$\frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-3x)} + \frac{E}{(1-3x)^2}$$

for some *fixed* constants A, B, C, D, E. The first term will have a partial fraction decomposition $\frac{F}{1-3x} + \frac{G}{(1-3x)^2}$, but now the constants F and G will depend on y_0 and y_1. When we finally use the binomial theorem to expand everything in a power series we will get a coefficient of x^n of the form

$$y_n = (a + bn) \cdot 1^n + c \cdot 2^n + (d + en) \cdot 3^n,$$

where a, b, c will be *fixed* constans, but d and e will depend on F and G, hence ultimately on y_0 and y_1. This is the general form of the solution, and to determine the exact solution requires knowing y_0 and y_1 and inserting these to solve for d and e.

17. Let $F_k(x) := \sum_{n=0}^{\infty} \lambda(n,\, k)\, x^k$.

Step 1: $k = 0$. We have $\lambda(0,\, 0) = 1$ and $\lambda(0,\, k) = 0$ for all $k \geq 1$, hence $F_0(x) = 1$.

Step 2: $k = 1$. We have $\lambda(n,\, 1) = n$, hence

$$F_1(x) = \sum_{n=0}^{\infty} nx^n \overset{(0.13)}{=} \frac{x}{(1-x)^2}. \tag{0.14}$$

Step 3: Suppose $k \geq 2$. Note that $\lambda(0,\, k) = \lambda(1,\, k) = 0$. Now, on the one hand,

$$xF_k(x) = \sum_{n=0}^{\infty} \lambda(n,\, k)x^{n+1} = \lambda(0,\, k)x + \sum_{n=2}^{\infty} \lambda(n-1,\, k)x^n = \sum_{n=2}^{\infty} \lambda(n-1,\, k)x^n, \tag{0.15}$$

$$x^2 F_{k-1}(x) = \sum_{n=0}^{\infty} \lambda(n,\, k-1)x^{n+2} = \sum_{n=2}^{\infty} \lambda(n-2,\, k-1)x^n. \tag{0.16}$$

On the other hand, using the recursion derived in Ex. 19.7.11,

$$F_k(x) = \lambda(0,\, k) + \lambda(1,\, k)x + \sum_{n=2}^{\infty} \lambda(n,\, k)x^n =$$

$$= 0 + 0 + \sum_{n=2}^{\infty} [\lambda(n-2,\, k-1) + \lambda(n-1,\, k)]x^n \overset{(0.15)+(0.16)}{=} xF_k(x) + x^2 F_{k-1}(x),$$

which implies that, for all $k \geq 2$,

$$F_k(x) = \left(\frac{x^2}{1-x}\right) F_{k-1}(x).$$

Together with (0.14), it is easily deduced that, for all $k \geq 1$,

$$F_k(x) = \frac{x^{2k-1}}{(1-x)^{k+1}}.$$

Finally, we expand the RHS of this using the binomial theorem to obtain

$$F_k(x) = x^{2k-1} \sum_{m=0}^{\infty} \binom{(k+1)+m-1}{m} x^m = x^{2k-1} \sum_{m=0}^{\infty} \binom{k+m}{k} x^m.$$

The coefficient of x^n is zero for all $n < 2k-1$ and $\binom{k+(n-(2k-1))}{k} = \binom{n-k+1}{k}$ otherwise. In other words, for all $k \geq 1$,

$$\lambda(n,\, k) = \begin{cases} 0, & \text{if } n < 2k-1, \\ \binom{n-k+1}{k}, & \text{if } n \geq 2k-1, \end{cases} \quad \text{v.s.v.}$$

18. Let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 1$ and the recursion, we can write

$$\begin{aligned} U(x) = 1 + \sum_{n=1}^{\infty} u_n x^n &= 1 + x \left(\sum_{n=0}^{\infty} u_{n+1}\, x^n \right) \\ &= 1 + x \left(3 \sum_{n=0}^{\infty} u_n x^n + \frac{1}{2} \sum_{n=0}^{\infty} 2^n x^n \right) = \\ &= 1 + x \left(3U(x) + \frac{1}{2(1-2x)} \right) \Rightarrow \\ \Rightarrow (1-3x)U(x) &= 1 + \frac{x}{2(1-2x)} \Rightarrow \\ \cdots \Rightarrow U(x) &= \frac{2-3x}{2(1-2x)((1-3x)}. \end{aligned}$$

Next, we make a partial fraction decomposition,

$$\begin{aligned} \frac{2-3x}{2(1-2x)(1-3x)} &= \frac{A}{1-2x} + \frac{B}{1-3x} \\ \Rightarrow 2 - 3x &= 2A(1-3x) + 2B(1-2x) \\ \Rightarrow 2A + 2B &= 2, \quad 6A + 4B = 3, \end{aligned}$$

which yields $A = -1/2$, $B = 3/2$. Hence,

$$U(x) = \frac{-1/2}{1-2x} + \frac{3/2}{1-3x}.$$

Finally, using the Binomial Theorem,

$$\begin{aligned} U(x) &= -\frac{1}{2} \sum_{n=0}^{\infty} 2^n x^n + \frac{3}{2} \sum_{n=0}^{\infty} 3^n \cdot x^n \\ \Rightarrow u_n &= \frac{3^{n+1}}{2} - 2^{n-1}. \end{aligned}$$

19. See Homework 1.

20. The relevant result from Ex. 12.7.10 is

$$q_n = \sum_{k=0}^{n-1} \binom{n-1}{k} q_k. \tag{0.17}$$

Let me denote the generating function by $Q(x)$, I don't want to write a tilde everywhere. We will first prove that

$$Q'(x) = e^x Q(x). \tag{0.18}$$

To achieve this, first use the definition of EGF:

$$Q(x) = \sum_{n=0}^{\infty} \frac{q_n}{n!} x^n.$$

Differentiating both sides w.r.t. x gives

$$Q'(x) = \sum_{n=1}^{\infty} \frac{q_n}{(n-1)!} x^{n-1}.$$

Now substitute for q_n using (0.17):

$$\begin{aligned} Q'(x) &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \left(\sum_{k=0}^{n-1} \binom{n-1}{k} q_k \right) = \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \left(\sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} q_k \right) = \\ &\overset{(n-1)! \text{ cancels}}{=} \sum_{n=1}^{\infty} x^{n-1} \left(\sum_{k=0}^{n-1} \frac{q_k}{k!(n-1-k)!} \right). \end{aligned}$$

Now comes the trick: interchange the order of summation above. Staring at it gives

$$\begin{aligned} Q'(x) &= \sum_{k=0}^{\infty} \frac{q_k}{k!} \left(\sum_{n=k+1}^{\infty} \frac{x^{n-1}}{(n-1-k)!} \right) = \\ &= \sum_{k=0}^{\infty} \frac{q_k}{k!} x^k \left(\sum_{n=k+1}^{\infty} \frac{x^{n-1-k}}{(n-1-k)!} \right) = \\ &\overset{m:=n-1-k}{=} \sum_{k=0}^{\infty} \frac{q_k}{k!} x^k \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} \right) = \\ &= e^x \left(\sum_{k=0}^{\infty} \frac{q_k}{k!} x^k \right) = e^x Q(x), \end{aligned}$$

which establishes (0.18). Now rewrite (0.18) as

$$\begin{aligned} \frac{Q'(x)}{Q(x)} &= e^x = \frac{d}{dx}(\ln Q(x)) \\ \Rightarrow \ln Q(x) &= \int e^x \, dx + C = e^x + C \\ \Rightarrow Q(x) &= e^{e^x + C} = C_1 \cdot e^{e^x}, \end{aligned}$$

for some constant C_1. To determine C_1, we insert $x = 0$. The RHS is $C_1 \cdot e^{e^0} = C_1 \cdot e^1 = C_1 \cdot e$. On the other hand, inserting $x = 0$ into the EGF gives the LHS as $Q(0) = q_0$. But $q_0 = 1$: one must define $q_0 = 1$ for (0.17) to be correct, since if we set $n = 1$ then (0.17) says that $q_1 = q_0$, and $q_1 = 1$ since there is always exactly one way to partition a set into one part.

Hence, $1 = C_1 \cdot e$, so $C_1 = e^{-1}$ and $Q(x) = e^{-1} e^{e^x} = e^{e^x - 1}$, v.s.v.