EXERCISES: CHAPTER 5

Section 5.1 (The concept of a function)

1. Let U denote the set of citizens of the state of Utopia. Which of the following statements correctly specify a function from U to U ? (Any assumptions you make about the Utopian civilisation should be stated explicitly).

(i) $f(x)$ is the mother of x.
(ii) $g(x)$ is the daughter of x.
(iii) $h(x)$ is the wife of x.

2. Write down the values $s(1)$, $s(2)$, $s(3)$, $s(4)$, $s(5)$, $s(6)$ of the function (sequence) defined by the rules

$$s(1) = 1, \quad s(2) = 2, \quad s(n+1) = 2s(n) - s(n-1) \ \ \forall\, n \geq 2.$$

Make a conjecture about a formula for $s(n)$ and try to prove it by using the principle of induction.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by the rule

$f(n)$ is the number of primes p such that $p \leq n$.

Make a table of the values of $f(n)$ for $1 \leq n \leq 50$. Is there a formula for $f(n)$?

Section 5.2 (Surjections, injections, bijections)

1. Suppose that the sets A and B are $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$, and that the functions $p : A \to B$, $q : B \to B$, $r : B \to A$ are defined as follows:

$$\begin{aligned} p(a) &= 3, & q(1) &= 3, & r(1) &= a, \\ p(b) &= 4, & q(2) &= 5, & r(2) &= c, \\ p(c) &= 2, & q(3) &= 1, & r(3) &= b, \\ p(d) &= 4, & q(4) &= 4, & r(4) &= a, \\ & & q(5) &= 2, & r(5) &= d. \end{aligned}$$

Which of these functions are surjections, which are injections and which are bijections ?

2. Which of the following functions from $\mathbb{N}$ to $\mathbb{N}$ are surjections, which of them are injections and which are bijections ?

$$f(n) = n^3, \quad g(n) = n + 3, \quad h(n) = \begin{cases} n+1, & \text{if } n \text{ is odd;} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

3. The function u from $\mathbb{N}$ to $\mathbb{N}$ is defined recursively by the rules

$$u(1) = 1, \quad u(n+1) = \begin{cases} \frac{1}{2}u(n), & \text{if } u(n) \text{ is even;} \\ 5u(n)+1, & \text{otherwise.} \end{cases}$$

Show that u is neither an injection nor a surjection.

Section 5.3 (Composition of functions)

1. The functions s and t from $\mathbb{N}$ to $\mathbb{N}$ are defined by

$$s(x) = x + 1, \quad t(x) = 2x \quad \forall\, x \in \mathbb{N}.$$

Show that $t \circ s$ and $s \circ t$ are different functions.

2. Let $X = \{1, 2, 3, 4, 5\}$ and let $f : X \to X$ be the function defined by

$$f(1) = 2, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 4, \quad f(5) = 4.$$

Show that $f \circ f = f$. Find another function $g \neq f$ such that $g \circ f = f \circ g = f$.

3. Suppose that f, g and h are functions such that the composite $h \circ (g \circ f)$ is defined. Show that $(h \circ g) \circ f$ is also defined and that $h \circ (g \circ f) = (h \circ g) \circ f$.

Section 5.4 (Bijections and inverse functions)

1. Construct bijections $s, t : S \to S$, where S is the set $\{1, 2, 3\}$, having the following properties ($i : S \to S$ is the identity function):

(i) $s \circ s = i$, $s \neq i$.
(ii) $t \circ t \circ t = i$, $t \neq i$.

Describe the inverse functions s^{-1} and t^{-1}, in terms of s and t.

2. Let S be as in Ex. 1. How many different bijections f from S to S are there, and how many of them satisfy $f = f^{-1}$?

3. Prove that if f and g are bijections and $g \circ f$ is defined, then it is also a bijection with inverse $f^{-1} \circ g^{-1}$.

4. The function $f : X \to Y$ is said to have a *left inverse* $l : Y \to X$ if $l \circ f$ is the identity on X. Show that

(i) if f has a left inverse then it is an injection
(ii) if f is an injection then it has a left inverse.

Review Section 5.5

1. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined recursively by the rules

$$f(1) = 7, \quad f(n+1) = \begin{cases} 3f(n) + 1, & \text{if } f(n) \text{ is odd;} \\ \frac{1}{2}f(n), & \text{if } f(n) \text{ is even.} \end{cases}$$

Show that f is neither a surjection nor an injection.

2. Suppose that A is any set and $f : A \to A$ is a function satisfying the condition

$$f(f(x)) = x \quad \forall\, x \in A.$$

Show that f is a bijection.

3. Explain carefully why the function $f : \mathbb{N} \to \mathbb{N}$ defined by the rule $f(n) = n^2 + 1$ is an injection but not a surjection.

4. Is it possible to construct a function $f : \mathbb{N} \to \mathbb{N}$ that is a surjection but not an injection ?

5. In this question, we shall refer to the following subsets of $\mathbb{N}$ as *blocks*:

$$\{1, 2, 4\},\ \ \{2, 3, 5\},\ \ \{3, 4, 6\},\ \ \{4, 5, 7\},\ \ \{1, 5, 6\},\ \ \{2, 6, 7\},\ \ \{1, 3, 7\}.$$

Let X denote the set of pairs (x, y) where x and y are numbers between 1 and 7. Given $(x, y) \in X$ let

$$f(x, y) = \begin{cases} z, & \text{if } x \neq y \text{ and } \{x, y, z\} \text{ is a block,} \\ x, & \text{if } x = y. \end{cases}$$

Is f a function ? Is it an injection ?

6. Formulate and prove results about a *right inverse* of a function $f : X \to Y$ analogous to those in Exercise 5.4.4.

7. Let Y denote the set of natural numbers all of whose decimal digits are 7. Construct a bijection $f : \mathbb{N} \to Y$.