ANSWERS: CHAPTER 5

Section 5.1

1. This is a pretty stupid exercise, I think, but let me try and make some sensible remarks on its various parts.

(i) Assuming the civilisation U has not found an alternative to sexual reproduction and has been around for more than one generation, then any currently living member of U will have exactly *one* mother (and one father). As such, the function f is well-defined, though its domain will not be all of U. For example, the mothers of some currently living members of U may be dead. Even if we include in U all members of the species who have ever lived, then there will still be some member without a mother (i.e.: their mother would be regarded as a member of a different species), assuming Darwinian evolution works in U as it does on Earth.
(ii) g is not defined for those members of U who either have no daughters or who have more than one daughter. Hence it's only a function if the domain is restricted to those members of U who have *exactly* one daughter.
(iii) Assuming neither polygamy nor same-sex marriage is allowed in U, then h is a function if its domain is restricted to the subset of married males.

2. $s(n) = n$ for all $n \in \mathbb{N}$.

3.

$$\begin{gathered} f(1)=0,\ \ f(2)=1,\ \ f(3)=f(4)=2,\ \ f(5)=f(6)=3, \\ f(7)=\cdots=f(10)=4,\ \ f(11)=f(12)=5,\ \ f(13)=\cdots=f(16)=6, \\ f(17)=f(18)=7,\ \ f(19)=\cdots=f(22)=8,\ \ f(23)=\cdots=f(28)=9, \\ f(29)=f(30)=10,\ \ f(31)=\cdots=f(36)=11,\ \ f(37)=\cdots=f(40)=12, \\ f(41)=f(42)=13,\ \ f(43)=\cdots=f(46)=14,\ \ f(47)=\cdots=f(50)=15. \end{gathered}$$

There is no formula for $f(n)$, but a famous theorem called the *Prime Number Theorem* states that $\lim_{n\to\infty} f(n)/\frac{n}{\ln n} = 1$.

Section 5.2

1. r is a surjection and q is a bijection.

2. f and g are both injective but neither is surjective. h is bijective.

3. One can compute one by one the first eight values of u as follows:

$$u(1) = 1,\ \ u(2) = 6,\ \ u(3) = 3,\ \ u(4) = 16,\ \ u(5) = 8,\ \ u(6) = 4,\ \ u(7) = 2,\ \ u(8) = 1.$$

The fact that $u(8) = u(1)$ already implies that u is not injective. Moreover, the definition of u says that $u(n+1)$ is determined by $u(n)$. Hence, once a value is repeated, the whole sequence of values up to that point will be repeated forever. In other words, $u(7k + l) = u(l)$ for all $k \in \mathbb{N}_0$ and all $1 \leq l \leq 7$ (the function u is said to be *periodic*, with period 7). In particular, u only takes on 7 distinct values, hence is a long way from being surjective.

Section 5.3

1. $(t \circ s)(x) = 2x + 2$ whereas $(s \circ t)(x) = 2x + 1$.

2. One can take $g = i$, the identity function. More generally, any function g with the following properties will work:

$$g(1) \in \{1,\, 2\},\ \ g(2) = 2,\ \ g(3) \in \{3,\, 4,\, 5\},\ \ g(4) = 4,\ \ g(5) \in \{3,\, 4,\, 5\}.$$

3. This is the well-known fact that composition of functions is *associative*.

Section 5.4

1. **(i)** There are three such functions $s_1,\, s_2,\, s_3$ given by

$$(1,\, 2,\, 3) \overset{s_1}{\to} (2,\, 1,\, 3), \quad (1,\, 2,\, 3) \overset{s_2}{\to} (3,\, 2,\, 1), \quad (1,\, 2,\, 3) \overset{s_3}{\to} (1,\, 3,\, 2).$$

(ii) There are two such functions $t_1,\, t_2$ given by

$$(1,\, 2,\, 3) \overset{t_1}{\to} (2,\, 3,\, 1), \quad (1,\, 2,\, 3) \overset{t_2}{\to} (3,\, 1,\, 2).$$

One has $s^{-1} = s$ and $t^{-1} = t \circ t$. In fact, $s_i^{-1} = s_i$ for each $i = 1,\, 2,\, 3$, while $t_1^{-1} = t_2$ and $t_2^{-1} = t_1$.

2. There are in total $3! = 6$ permutations of S, the five given above and the identity. Four are their own inverses.

3. Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$. In other words, $g(f(x_1)) = g(f(x_2))$. Since g is injective, it follows that $f(x_1) = f(x_2)$. And since f is injective, it follows in turn that $x_1 = x_2$. This proves that $g \circ f$ is injective.

Suppose $f : X \to Y$ and $g : Y \to Z$. Let z be any element of Z. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then $(g \circ f)(x) = z$. This proves that $g \circ f$ is surjective.

Finally, the fact that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ follows from associativity of composition, since

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ i_Y \circ f = f^{-1} \circ f = i_X, \quad \text{v.s.v.}$$

4. Similar reasoning to the previous exercise.

Review Section 5.5

1. The first 16 values taken on by f are

$$7,\ 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1,\ 4.$$

Since $f(n)$ determines $f(n+1)$, the values $4,\ 2,\ 1$ will now be repeated indefinitely, thus f is neither injective nor surjective.

2. Suppose f is not injective. Then there must exist $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. But then we'd have $x_1 = f(f(x_1)) = f(f(x_2)) = x_2$, contradiction. So f is injective.

Let $a \in A$. Then $f(f(a)) = a$, i.e.: the element $f(a)$, whatever that is, gets mapped by f to a. Thus f is also surjective.

3. It's a strictly increasing function, hence injective. It's far from surjective since there are huge gaps in its range. For example, $f(1) = 2$ and, since f is strictly increasing, it never takes on the value 1.

4. Yes, for example $f(x) = \lceil \frac{x}{2} \rceil$ will take on every positive integer value exactly twice.

5. Denote $\mathbb{N}_7 := \{1,\ 2,\ \ldots,\ 7\}$. Yes, $f : X \to \mathbb{N}_7$ is a function defined on the whole of X, since for each pair $x,\ y$ of distinct elements of $\mathbb{N}_7$ there is exactly one $z \in \mathbb{N}_7$ such that $\{x,\ y,\ z\}$ is a block. Explicitly, $f(x,\ y) = f(y,\ x)$ and

$$\begin{array}{llllll}
f(1,\,2)=4, & f(1,\,3)=7, & f(1,\,4)=2, & f(1,\,5)=6, & f(1,\,6)=5, & f(1,\,7)=3, \\
 & f(2,\,3)=5, & f(2,\,4)=1, & f(2,\,5)=3, & f(2,\,6)=7, & f(2,\,7)=6, \\
 & & f(3,\,4)=6, & f(3,\,5)=2, & f(3,\,6)=4, & f(3,\,7)=1, \\
 & & & f(4,\,5)=7, & f(4,\,6)=3, & f(4,\,7)=5, \\
 & & & & f(5,\,6)=1, & f(5,\,7)=4, \\
 & & & & & f(6,\,7)=2.
\end{array}$$

f is not injective, as we can see from the above list. Indeed, since $|X| = 7^2 = 49$ and $|\mathbb{N}_7| = 7$, there is no injective function from X to $\mathbb{N}_7$.

6. If $f : X \to Y$ is a function, then a function $r : Y \to X$ is called a *right inverse* for f if $f \circ r$ is the identity function on Y.

The theorem is that f has a right inverse if and only if it is surjective. For if that is so then we can, for each $y \in Y$, find $x \in X$ such that $f(x) = y$. Then define $r(y) = x$ (note that the choice of r is not unique unless f is also injective).

7. For example, let $f(n)$ be the unique n-digit number in Y.