EXERCISES: CHAPTER 6

Section 6.1 (Counting as a bijection)

1. In each of the following cases write down a formula for a bijection $f : \{1, 2, \ldots, m\} \to X$, for an appropriate value of m:

(a) $X = \{2, 4, 6, 8, 10\}$.
(b) $X = \{3, 8, 13, 18, 23, 28\}$.
(c) $X = \{10, 17, 26, 37, 50, 65, 82, 101\}$.
(d) $X = \{k \in \mathbb{N} :$ the kth day of this month is a Monday$\}$.

2. Describe any difficulties that might arise in using the "child's method of pointing and counting aloud" to determine the sizes of the following types of sets:

(a) The set of sheep in a field.
(b) The set of your own ancestors.
(c) The set of even natural numbers.
(d) The empty set.

Section 6.2 (The size of a set)

1. Give an example of a function from $X = \{1, 2, 3\}$ to $Y = \{0, 1\}$. Let F denote the set of all functions $f : X \to Y$. Show that $|F| = 8$, by listing the members of F, calling them $f_1, f_2, \ldots, f_8$.

2. If $|X| = m$ and $|Y| = 2$, what is the size of the set F of all functions $f : X \to Y$?

3. Four new students have to be assigned to a tutor. There are seven possible tutors and none of them will accept more than one student. In how many ways can the assignment be carried out ?

Section 6.3 (A counting problem)

1. A very basic "counting principle", which we shall discuss in more detail in Chapter 10, concerns the relationship between $|A|$, $|B|$, $|A \cup B|$ and $|A \cap B|$. By looking at simple examples, try to formulate this relationship in general terms.

Section 6.4 (Some applications of the pigeonhole principle)

1. A blindfolded man has a heap of 10 grey socks and 10 brown socks in a drawer. How many must he select in order to guarantee that, among them, there is a matching pair ?

2. Let S be any set of 12 natural numbers. Show that S must contain two distinct numbers $s_1,\ s_2$ such that $s_1 - s_2$ is a multiple of 11.

3. Let X be a subset of $\{1,\ 2,\ \ldots,\ 2n\}$ and let Y be the set of odd numbers $\{1,\ 3,\ \ldots,\ 2n-1\}$. Define a function $f : X \to Y$ by the rule

$f(x) =$ the greatest member of Y that exactly divides x.

Show that if $|X| \geq n+1$ then f is not an injection, and deduce that in this case X contains distinct numbers x_1 and x_2 such that x_1 is a multiple of x_2.

4. With the notation as in Ex. 3, show that it is possible to find a subset X of $\{1,\ 2,\ \ldots,\ 2n\}$ which has n members and is such that no member of X is a multiple of any other.

5. Five professors of sociology wish to build houses on an island in the form of an equilateral triangle with sides of length 2 km. Is it possible for them to find sites so that each house is over 1 km from the others ? (HINT: Divide the triangle into 4 regions).

Review Section 6.7

1. In each of the following cases, either construct a bijection from the set $\mathbb{N}$ to the set Y or explain why such a bijection cannot exist.

(a) $Y =$ set of natural numbers that are multiples of 5.
(b) $Y =$ set of natural numbers that are less than 2000.
(c) $Y =$ set of natural numbers for which all the decimal digits are different.

2. Show that if X is a finite set and the function $g : X \to X$ is an injection, then g is a bijection.

3. Show that if X is a finite set and $f : X \to X$ is a surjection, then f is a bijection.

4. Prove that if any 10 points are chosen within an equilateral triangle of side-length 1, then there are two of them whose distance apart is at most $\frac{1}{3}$.

5. How many points must be chosen inside a square of side-length 2 in order to ensure that at least one pair are not more than distance $\sqrt{2}$ apart ?

6. Show that in any set of 172 natural numbers there must be a pair whose difference is divisible by 171. Is the result true if the word "difference" is replaced by "sum" ?

7. A golfer has d days to prepare for a tournament and must practice by playing at least one round each day. In order to avoid staleness he should not play more than m rounds altogether. Show that if r satisfies $1 \leq r \leq 2d - m - 1$, then there is a sequence of consecutive days during which he plays exactly r rounds.