ANSWERS: CHAPTER 6

Section 6.1

1. (a) $m = 5$. Take $f(x) = 2x$ for all $x = 1, 2, \ldots, m$.
(b) $m = 6$. Take $f(x) = -2 + 5x$ for all $x = 1, 2, \ldots, m$.
(c) $m = 8$. Take $f(x) = (x+2)^2 + 1$ for all $x = 1, 2, \ldots, m$.
(d) At the time of writing it is March 2016, and the Mondays this month were on the 7th, 14th, 21st and 28th. Thus $m = 4$ and take $f(x) = 7x$ for all $x = 1, 2, 3, 4$.

2. Another stupid question, but I will try to make some semi-intelligent comments:

(a) The sheep move around so it might be hard to keep track of who we've already counted.
(b) For one, there's no way you can trace back all your ancestors.
(c) It's an infinite set, so you'll never be finished counting.
(d) Well, there's nothing to count.

Section 6.2

1. Say $f_2(1) = f_2(2) = 0$, $f_2(3) = 1$. The full list is

$$\begin{aligned}
&f_1(1) = 0, \quad f_1(2) = 0, \quad f_1(3) = 0,\\
&f_2(1) = 0, \quad f_2(2) = 0, \quad f_2(3) = 1,\\
&f_3(1) = 0, \quad f_3(2) = 1, \quad f_3(3) = 0,\\
&f_4(1) = 0, \quad f_4(2) = 1, \quad f_4(3) = 1,\\
&f_5(1) = 1, \quad f_5(2) = 0, \quad f_5(3) = 0,\\
&f_6(1) = 1, \quad f_6(2) = 0, \quad f_6(3) = 1,\\
&f_7(1) = 1, \quad f_7(2) = 1, \quad f_7(3) = 0,\\
&f_8(1) = 1, \quad f_8(2) = 1, \quad f_8(3) = 1.
\end{aligned}$$

2. There are 2^m such functions, by the multiplication principle. Note that, as in Ex. 1 above, one can set up an explicit 1-1 correspondence between these 2^m functions and the set of all m-digit binary numbers.

3. Let X be the set of students and Y the set of tutors. Since each tutor will take at most one student, an assignment of students to tutors is just an injective function from X to Y. Hence, we seek the cardinality of the set F of injective functions $f : X \to Y$. As discussed in lectures, $|F| = P(7, 4) = 840$.

Section 6.3

1. $|A \cup B| = |A| + |B| - |A \cap B|$. See the lecture notes on the Inclusion-Exclusion principle.

Section 6.4

1. By the Pigeonhole Principle, he must select at least 3 socks to be guaranteed to have at least 2 in the same colour.

2. Let $T = \{0, 1, \ldots, 10\}$. Define a function $r : S \to T$ by

$r(s) =$ the remainder left by s upon division by 11.

Since $|S| = 12$ and $|T| = 11$, the function r is not injective, i.e.: there exist $s_1 \neq s_2$ such that $r(s_1) = r(s_2)$. But then $s_1 - s_2$ will be a multiple of 11.

3. $|Y| = n$, thus no function $f : X \to Y$ can be injective if $|X| \geq n + 1$. Suppose $f(x_1) = f(x_2) = d$ say. Thus d is odd, $d|x_1$, $d|x_2$ and no odd number greater than d divides both x_1 and x_2. It follows that the prime factoridations of x_1 and x_2 differ only in powers of 2, i.e.: there exist non-negative integers $i \neq j$ such that $x_1 = 2^i \cdot d$ and $x_2 = 2^j \cdot d$. If $i > j$ then x_2 divides x_1 and vice versa.

4. Take $X = \{n + 1, n + 2, \ldots, 2n\}$. By the way, once upon a time I wrote a research paper on a generalisation of this problem, see

`http://www.math.chalmers.se/~hegarty/k-primitive_integers_final.pdf`

5. See Homework 1.

Review Section 6.7

1. The important point here is that there is a bijection from $\mathbb{N}$ to any infinite subset of itself. For the elements of such an infinite subset Y can be listed in increasing order as $Y = \{y_1 < y_2 < y_3 < \ldots\}$ and then a bijection from $\mathbb{N}$ is obtained by mapping n to y_n for all $n \in \mathbb{N}$.

Thus bijections exist in cases **(a)** and **(c)**. In Case **(a)** we can easily give an explicit example, namely $f(n) = 5n$. It is harder to give an explicit example in Case **(c)**.

In Case **(b)** there is no bijection since Y is a finite set.

2. The underlying point is the simple one that, if A is a finite set and B is a subset of A such that $|B| = |A|$, then $B = A$. Here one would take $A = X$ and $B = g(X)$.

3. The underlying point here is the simple one that, if A is a finite set and B is a subset of A such that $|B| < |A|$, then $B \neq A$. If f were not injective, then $|X| > |f(X)|$ would hold and so $f(X)$ could not be all of X, i.e.: f could not be surjective.

4. Similar to Ex. 6.4.5. See Homework 1.

5. Similar to Ex. 6.4.5. See Homework 1.

6. For "difference" the reasoning is completely analogous to Ex. 6.4.2. For "sum" the result is completely false, no matter how many numbers are in our set S. For example, if every number in S leaves a remainder of 1 upon division by 171, i.e.: if S is a subset of $\{1 + 171n : n \in \mathbb{N}_0\}$, then the sum of any two elements of S will leave a remainder of two, not zero.

7. See Homework 1.