First Lecture: 21/3

The first part of the course is concerned with what is often called *Enumerative Combinatorics* which, informally, is the art of counting. In general terms one is interested in "clever" means of counting the sizes of finite sets, or at least estimating them - one is usually dealing with large sets, or a sequence $(A_n)_{n=1}^{\infty}$ of sets and then interested in estimating $|A_n|$ as $n \to \infty$.

Two very basic principles underlie a very common method of reasoning:

Addition Principle. *Let A and B be two finite, disjoint sets (i.e.: $A \cap B = \phi$). Then*

$$|A \cup B| = |A| + |B|. \tag{1.1}$$

More generally, if $A_1, A_2, \ldots, A_k$ are finite, pairwise disjoint sets, i.e.: $A_i \cap A_j = \phi$ for all $i \neq j$, then

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|. \tag{1.2}$$

Recall that the *Cartesian product* of two sets A and B is defined and denoted as

$$A \times B = \{(a, b) : a \in A, b \in B\}. \tag{1.3}$$

Multiplication Principle. *Let A and B be two finite sets. Then*

$$|A \times B| = |A| \times |B|. \tag{1.4}$$

The above two principles can be reformulated in various ways, each of which provides its own insights. One viewpoint which is particularly useful for counting problems is the following:

Addition Principle (2nd formulation). *Let A and B be two finite, disjoint sets. The number of ways to choose an element from* ***either*** *A* ***or*** *B is $|A| + |B|$.*

Multiplication Principle (2nd formulation). *Let A and B be two finite, disjoint sets. The number of ways to choose* ***both*** *an element from A* ***and*** *an element from B is $|A| \times |B|$.*

Example 1.1. Let A and B be any two sets. One denotes by B^A the set of all possible functions from A to B, i.e.:

$$B^A = \{f \mid f : A \to B \text{ is a function}\}. \tag{1.5}$$

Now suppose $|A| = m$ and $|B| = n$. Then $|B^A| = n^m$ (which also explains the notation). For if $A = \{a_1, a_2, \ldots, a_m\}$ then a function is given by an m-tuple $(f(a_1), f(a_2), \ldots, f(a_m))$, which can be an arbitrary m-tuple of elements of B, i.e.: an arbitrary element of the m-fold Cartesian product of B with itself (usually denoted B^m). By the multiplication principle, this product has size $|B|^m = n^m$, v.s.v.

Recall that a function $f : A \to B$ is said to be *injective* if $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$ are distinct elements of A.

Example 1.2. Let us take Example 1 a step further and ask for the number of injective functions from A to B. Clearly the number of such functions is zero if $m > n$.[1] Otherwise, such a function is once again given by an m-tuple $(f(a_1),\, f(a_2),\, \dots,\, f(a_m))$, but now the coordinates must be *distinct* elements of B. To determine the number of possibilities we reason as for the multiplication principle. There are $|B| = n$ choices for the first coordinate. Now one element of B is used up, so there are $n-1$ choices for the second coordinate. Then $n-2$ choices for the third coordinate and so on, until finally there are $n-m+1$ choices for the m:th coordinate. We must make a choice of *every* coordinate to determine the function f, so the multiplication principle applies.

We have proven that the number of injective functions from an m-element set to an n-element set is given by

$$P(n,\, m) = n(n-1)(n-2)\dots(n-m+1) = \prod_{i=1}^{m}(n-i+1). \tag{1.6}$$

Note in particular that $P(n,\, n) = n!$. As a special case, let $A = B$. An injective function from a finite set to itself is also *surjective*[2] and hence *bijective*. A bijection from a set to itself is usually termed a *permutation* of the set. Hence, a corollary to Example 1.2 is the following

Proposition 1.3. *Let A be a finite set with n elements. The number of permutations of A is $P(n\,n) = n!$.*

Notation. The default choice of an n-element set is $\{1,\, 2,\, \dots,\, n\}$. One denotes by S_n the set of all permutations of this set. It is called the *symmetric group of order* n, which refers to the fact that this set has the algebraic structure of a group. We will not say more on this for the moment, however (as I am not going to assume everyone knows what a “group” is). For now, it suffices to know the notation and terminology.

Note that we may describe the quantity $P(n,\, m)$ as the number of ways to choose m distinct elements from an n-element set, where the *order* in which the m elements are chosen is important.

We let $C(n,\, m)$ denote the corresponding quantity where the order of choice is NOT important, hence where it only matters which m elements are chosen. It is easy to see that

$$C(n,\, m) = \frac{P(n,\, m)}{m!}, \tag{1.7}$$

since for every choice of m elements there are $m!$ ways to order (i.e.: permute) them.

Notation. A more common notation for $C(n,\, m)$ is $\binom{n}{m}$. In words, “n choose m” (på svenska “n över m”).

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[1] This is a special case of the so-called *Pigeonhole principle*, also called the *Dirichlet box principle*, which we will return to in Lecture xx.

[2] A function $f : A \to B$ is said to be surjective if, for each $b \in B$ there is at least one $a \in A$ such that $f(a) = b$. Note by the way that an injective function on an *infinite* set need not be surjective (think of an example !)

Proposition 1.4.

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{1.8}$$

Proof. We already know that

$$\binom{n}{m} = \frac{P(n,\, m)}{m!} = \frac{n(n-1)\dots(n-m+1)}{m!} \tag{1.9}$$

Now simply multiply the last expression above and below by $(n-m)!$ and note that the numerator then becomes $n!$. □

Note an immediate and useful consequence of formula (1.8):

Proposition 1.5.

$$\binom{n}{m} = \binom{n}{n-m}. \tag{1.10}$$

Proof. This follows immediately from the symmetry in the denominator of (1.8). Another way to understand (1.10) is to realise that the number of ways to choose m elements from n is the same as the number of ways to reject $n-m$ elements. □

Many counting problems involve use of both the addition and multiplication principles. We close this lecture with a typical example. To understand the example you need to know that a "stryktipsrad" involves a guess of the outcomes of each of 13 football matches. There are 3 possible outcomes for each match: 1, X or 2.

Example 1.6. Let us compute the number of ways to fill in a stryktipsrad such that one gets at least 10 results right. One imagines the results of all 13 games being known to some superior being who can see into the future. Let A_{10}, A_{11}, A_{12}, A_{13} denote, respectively, the number of ways to fill in the tipskupong so that one gets 10, 11, 12, 13 correct. Then we are interested in the size of the union of these sets. Note that the sets are pairwise disjoint (by definition) so, by the addition principle,

$$|A_{10} \cup A_{11} \cup A_{12} \cup A_{13}| = |A_{10}| + |A_{11}| + |A_{12}| + |A_{13}|. \tag{1.11}$$

Let us compute the size of A_{10} - the others are handled similarly. There are 13 results and 10 of these are correctly guessed. There are $\binom{13}{10}$ possible choices of the 10 correctly guessed results. This leaves 3 results which were incorrectly guessed and, for each of these, there are 2 ways to guess incorrectly. Now the multiplication principle applies, since we have to guess the results of *all* 13 matches. Hence,

$$|A_{10}| = \binom{13}{10} \times 2^3. \tag{1.12}$$

Similarly,

$$|A_{11}| = \binom{13}{11} \times 2^2, \quad |A_{12}| = \binom{13}{12} \times 2^1, \quad |A_{13}| = \binom{13}{13} \times 2^0. \tag{1.13}$$

Substituting into (1.11) and using (1.10) to simplify, we find that

$$|A_{10} \cup A_{11} \cup A_{12} \cup A_{13}| =$$
$$= \binom{13}{3} \times 2^3 + \binom{13}{2} \times 2^2 + \binom{13}{1} \times 2^1 + \binom{13}{0} \times 2^0 =$$
$$= \left(\frac{13 \times 12 \times 11}{1 \times 2 \times 3}\right) \times 8 + \left(\frac{13 \times 12}{1 \times 2}\right) \times 4 + 13 \times 2 + 1 \times 1 =$$
$$= 2288 + 312 + 26 + 1 = 2627.$$

So there are 2627 different ways to get at least 10 results right. Note that the total number of ways to fill in the tipskupong is 3^{13}, by the multiplication principle, since there are 3 ways to fill in each row. Hence, the *probability* of getting at least 10 results right (if you just guess totally at random) is $\frac{2627}{3^{13}} \approx \frac{1}{607}$. We remark that the betting companies usually pay out to everyone who gets at least 10 result correct. However, the payout is usually *much* less than "607 gånger pengarna", which reflects two salient facts: (i) they want to make a profit (ii) the punters are usually people with some knowledge of football, they are not guessing randomly.