Tenth Lecture: 25/4

Definition 10.1. Let $A \subseteq \mathbb{N}_0$. The *characteristic function* of A is the function $\chi_A : \mathbb{N}_0 \to \{0, 1\}$ defined by

$$\chi_A(n) = \begin{cases} 1, & \text{if } n \in A, \\ 0, & \text{if } n \notin A. \end{cases}$$

Definition 10.2. Let $A \subseteq \mathbb{N}_0$. The *generating function* of the set A , denoted $G_A(x)$, is defined to be the generating function of the sequence $(\chi_A(n))_{n=0}^{\infty}$. In other words,

$$G_A(x) = \sum_{n \in A} x^n. \tag{10.1}$$

Note that, if A is an infinite set, then the series converges when $|x| < 1$.

Proof of Theorem 9.12. The proof is by contradiction. Suppose there existed a set $A \subseteq \mathbb{N}_0$ such that $r_2(A, n) = t > 0$ for all $n \geq n_0$. Let $G(x) = G_A(x)$ be the generating function of A, as defined in (10.1). The generating function connects to the representation function as follows:

$$[G(x)]^2 + G(x^2) = 2 \cdot \sum_{n=0}^{\infty} r_2(A, n)\, x^n. \tag{10.2}$$

To see why this is so, observe that in the first term on the LHS, every representation of the form $n = a_1 + a_2$ is counted twice when $a_1 \neq a_2$ and once when $a_1 = a_2$. The second term on the LHS counts representations $n = a + a$ exactly once. Hence, in total on the LHS, every representation of an integer n as a sum of two elements of A is counted exactly twice, which is what the RHS says explicitly.

Now we are supposing that $r_2(A, n) = t$ for all $n \geq n_0$. Then (10.2) can be written as

$$[G(x)]^2 + G(x^2) = \sum_{n=0}^{n_0-1} r_2(A, n)\, x^n + 2t \cdot \sum_{n=n_0}^{\infty} x^n. \tag{10.3}$$

The first sum on the RHS of (10.3) is some polynomial in x, call it $P(x)$. The second sum is a geometric series, so has a simple formula. We thus find that

$$[G(x)]^2 + G(x^2) = P(x) + 2t \cdot \frac{x^{n_0}}{1-x}. \tag{10.4}$$

We obtain a contradiction by seeing what happens as $x \to -1^+$. First consider the LHS of (10.4). The term $G(x^2)$ converges towards $G((-1)^2) = G(1) = \sum_{n \in A} 1^n = \sum_{n \in A} 1 = |A| = +\infty$, since A must at the very least be an infinite set if it is to be a basis of finite order. The term $[G(x)]^2$ is always non-negative, simply because it's the square of something. Hence, the LHS of (10.4) tends to $+\infty$ as $x \to -1^+$. But the RHS heads towards some finite value, namely $P(-1) + (-1)^{n_0} \cdot t$. This contradiction completes the proof. $\square$

Example 10.3. Show that, in any group of 6 people, there must either be three mutual friends or three mutual strangers.

Solution: The above is the popular formulation of the basic observation of Ramsey theory. Here "friendship" stands for any symmetric relation between elements of a set. So the more abstract formulation of the problem is to show that, given any symmetric relation on a set of 6 elements, there must exist either a subset of three elements all of which are related, or a subset of three elements none of which are related.

The idea for the solution is a prototype of the method we'll use to prove Theorem 10.6 below. Isolate one of the six people, call him P. There are $5 = (2+2)+1$ other people present, which leaves two possibilities:

CASE 1: P has at least three friends. If none of P's friends know one another, then they form a group of at least three mutual strangers and we're done. Otherwise, some two of P's friends know each other, in which case these two together with P form a group of three mutual friends.

CASE 2: There are at least three strangers to P. If all these guys know one another, then they form a group of at least three mutual friends, and we're done. Otherwise, some pair of them are strangers, in which case these two, together with P, form a group of three mutual strangers. This completes the solution.

Remark 10.4. It is possible for the friendship relations in a group of 5 people to be such that there exists neither a subset of three mutual friends nor a subset of three mutual strangers. Indeed, there is only one way to make this work, up to a permutation of the 5 people. It is illustrated in Figure 10.1 on the homepage.

We are now ready to define Ramsey numbers, slightly informally in the language of friends and strangers. A more formal definition will be given in the next lecture.

Definition 10.5. Let $k, l \in \mathbb{N}_{\geq 2}$. The *Ramsey number* $R(k,\, l)$ is the smallest $n \in \mathbb{N}$ such that, in any group of n people, there must exist either a subset of k mutual friends or a subset of l mutual strangers.

From Example 10.3 and Remark 10.4 it follows that $R(3,\, 3) = 6$. The fundamental result proven by Frank Ramsey in the 1920s, whose proof follows next time, is the following:

Theorem 10.6. *The numbers $R(k,\, l)$ all exist, i.e.: are finite. In fact, we have*

$$R(k,\, l) = R(l,\, k), \tag{10.5}$$

$$R(k,\, 2) = R(2,\, k) = k, \tag{10.6}$$

and, in general,

$$R(k,\, l) \leq R(k-1,\, l) + R(k,\, l-1). \tag{10.7}$$