Eleventh Lecture: 27/4

Proof of Theorem 10.6. Note that the three statements (10.5)-(10.7) together imply that all the numbers $R(k, l)$ are finite, say by an induction on $k + l$.

Firstly, (10.5) is obvious since, in Definition 10.5, there is a symmetry between the concepts of "friendship" and its opposite, "strangership". Eq. (10.6) is equally obvious as it merely says that, if we have a group of l people, then either they are all mutual friends (resp. strangers) or at least one pair are strangers (resp. friends).

So we turn to (10.7). Here we are assuming that the numbers $R(k - 1, l)$ and $R(k, l - 1)$ are finite and wish to deduce an upper bound for $R(k, l)$. Let $n := R(k-1, l) + R(k, l-1)$. We must show that, in any group of n people, there exists either a subset of k mutual friends or a subset of l mutual strangers. Isolate any person in the group, call him P. There are $n - 1 = [R(k - 1, l) - 1] + [R(k, l - 1) - 1] + 1$ other people, so at least one of the following two cases must arise:

CASE 1 : P has at least $R(k - 1, l)$ friends. By definition of the Ramsey number it follows that, amongst P's friends, we can find either a subset of l mutual strangers - in which case we're done - or a subset of $k - 1$ mutual friends. In the latter instance, such a group together with P form a subset of k mutual friends.

CASE 2 : P is a stranger to at least $R(k, l - 1)$ others. By definition of the Ramsey number it follows that, amongst the strangers to P, we can find either a subset of k mutual friends - in which case we're done - or a subset of $l - 1$ mutual strangers. In the latter instance, such a group together with P form a subset of l mutual strangers. □

From Theorem 10.6 we can deduce explicit upper bounds for Ramsey numbers:

Corollary 11.1. *For every $k, l \geq 2$ we have*

$$R(k, l) \leq \binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}. \tag{11.1}$$

Proof. For $l = 2$ this reduces to

$$R(k, 2) \leq \binom{k}{k-1},$$

which is true, since both sides equal k, by (10.6). To establish the inequality in general, we proceed by induction on $k + l$. By (10.7), this easily reduces to showing that

$$\binom{k+l-2}{k-1} \geq \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1}.$$

And this is true, in fact we have equality again, by Pascal's identity for binomial coefficients (2.2). □

It is natural to consider the special (symmetric) case $k = l$. The numbers $R(k, k)$ are called *diagonal Ramsey numbers*. Here (11.1) becomes

$$R(k, k) \leq \binom{2k-2}{k-1}.$$

Using simply the fact that

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

(both sides count the number of subsets of an n-element set), it follows that

$$R(k,\, k) \leq 4^{k-1}.$$

A slightly better estimate can be got using Stirling's formula[1], but the important point is that the upper bound we obtain for $R(k,\, k)$ is exponential in k. A problem which Ramsey didn't solve, and which remained open for a few years after his untimely death in 1930 at the age of 26, was whether these diagonal Ramsey numbers really do grow exponentially. It turns out that they do[2], and the proof of this fact is perhaps the oldest application of what has become known as the *probabilistic method in combinatorics*. The formal result is

Theorem 11.2. *Let* $k \geq 3$*. If the positive integer* n *satisfies*

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1, \tag{11.2}$$

then $R(k,\, k) > n$.

Before proving this result, we show how it implies an exponential lower bound for diagonal Ramsey numbers:

Corollary 11.3.

$$R(k,\, k) \geq 2^{k/2} = (\sqrt{2})^k. \tag{11.3}$$

Proof of Corollary. From (10.6) it follows that (11.3) holds with equality for $k = 2$. So we may assume $k \geq 3$. It suffices to show that, if $n = 2^{k/2}$, then (11.2) is satisfied. Since $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} < \frac{n^k}{k!}$ and $\binom{k}{2} = \frac{k(k-1)}{2}$, the left-hand side of (11.2) will be less than

$$\frac{(2^{k/2})^k}{k!} \times 2^{1-\frac{k(k-1)}{2}} = \frac{2^{1+\frac{k}{2}}}{k!}.$$

So we just need to check that $2^{1+\frac{k}{2}} \leq k!$, for every $k \geq 3$. For $k = 3$ it says that $4\sqrt{2} \leq 6$, which is true. If we now replace k by $k+1$, then the LHS of the inequality is multiplied by $\sqrt{2}$ and the RHS is multiplied by $k+1$, which is certainly greater than $\sqrt{2}$, for any $k \geq 3$. Hence the inequality holds for all $k \geq 3$. □

Remark 11.4. It is another Fields-Medal level open problem to decide whether or not $\lim_{k\to\infty} R(k,\, k)^{1/k}$ exists and to determine the limit if it does exist. What we've said above implies that

$$\sqrt{2} \leq \liminf_{k\to\infty} R(k,\, k)^{1/k} \leq \limsup_{k\to\infty} R(k,\, k)^{1/k} \leq 4.$$

[1] $n! \sim n^n e^{-n}\sqrt{2\pi n}$

[2] Only a finite (and very small !) collection of Ramsey numbers $R(k,\, l)$, for $k,\, l \geq 3$, have been computed exactly. For example, we showed last time that $R(3, 3) = 6$. For an up-to-date list of known values, see `http://mathworld.wolfram.com/RamseyNumber.html`

Even today, nothing better than this is known. This is also an *Erdős prize problem*, apparently it was one of his personal favorites.

The proof of Theorem 11.2 will come next day. It uses ideas from (elementary) probability theory, which basically involve recasting the addition, multiplication and Inclusion-Exclusion principles in probabilistic language.

Notation 11.5. In probability theory, if A and B are events, then $A \vee B$ denotes the event "A or B", and $A \wedge B$ denotes the event "A and B". In other words, the event $A \vee B$ is said to have occurred if and only if at least one of A and B have done so, whereas the event $A \wedge B$ is said to have occurred if and only if both A and B have done so.

We start by translating the Addition Principle, as formulated in eqs. (1.1) and (1.2), into probabilistic language. Events A and B are said to be *mutually exclusive* if it is not possible for both to occur simoultaneously. Then the probabilistic version of (1.1) and (1.2) is:

Probabilistic Addition Principle. *If A and B are mutually exclusive events, then*

$$\mathbb{P}(A \vee B) = \mathbb{P}(A) + \mathbb{P}(B). \tag{11.4}$$

More generally, if $A_1, A_2, \ldots, A_k$ are pairwise mutually exclusive events, then

$$\mathbb{P}(A_1 \vee A_2 \vee \cdots \vee A_k) = \sum_{i=1}^{k} \mathbb{P}(A_i). \tag{11.5}$$

The Inclusion-Exclusion principle deals with sets which are not necessarily pairwise disjoint. Then the probabilistic version of (2.7) is as follows:

Probabilistic Inclusion-Exclusion Principle. *For any finite collection of events $A_1, A_2, \ldots, A_k$, one has*

$$\mathbb{P}(A_1 \vee \cdots \vee A_k) = \sum_{i=1}^{k} \mathbb{P}(A_i) - \sum_{i \neq j} \mathbb{P}(A_i \wedge A_j) + \sum_{i \neq j \neq k} \mathbb{P}(A_i \wedge A_j \wedge A_k) - \cdots + (-1)^{k-1} \mathbb{P}(A_1 \wedge \cdots \wedge A_k). \tag{11.6}$$

In our proof of Theorem 11.2 we will use an apparently very weak version of (11.6), which nevertheless has its own name because it often suffices for applications of probabilistic methods in combinatorics:

Union Bound. *For any finite collection of events $A_1, A_2, \ldots, A_k$, one has*

$$\mathbb{P}(A_1 \vee \cdots \vee A_k) \leq \sum_{i=1}^{k} \mathbb{P}(A_i). \tag{11.7}$$

Turning to the multiplication principle, one has the following recasting of (1.4):

Probabilistic Multiplication Principle. *If A and B are independent events, then*

$$\mathbb{P}(A \wedge B) = \mathbb{P}(A) \cdot \mathbb{P}(B). \tag{11.8}$$

More generally, if $A_1,\, A_2,\, \dots,\, A_k$ are independent events, then

$$\mathbb{P}(A_1 \wedge \cdots \wedge A_k) = \prod_{i=1}^{k} \mathbb{P}(A_i). \tag{11.9}$$

Remark 11.6. I'm actually cheating a bit here, since I haven't defined what I mean by *independent* events. Actually, to define this precsiely is a bit technical, so I don't want to go into it. For the application to come, I hope it will be obvious that the events considered *are* independent.

Finally for today, I want to recast the definition of Ramsey numbers in terms of so-called *edge colorings of the complete graph*, as it will be easier to formulate the proof of Theorem 11.2 in this terminology. So here are the requisite definitions. The basic graph concept will be the foundation for the rest of the course.

Definition 11.7. A *graph* G is a pair $(V,\, E)$, where V is a finite set and E is a subset of $\binom{V}{2}$, i.e.: of the collection of all pairs $\{v_1,\, v_2\}$, $v_1 \neq v_2$, of distinct elements of V.

The elements of V are called the *vertices* or *nodes* of the graph G and the elements of E are called its *edges*.

Example 11.8. For the graph in Figure 11.1 on the homepage one has

$$V = \{1,\, 2,\, 3,\, 4,\, 5\}, \quad E = \{\{1,\, 2\},\, \{1,\, 3\},\, \{1,\, 5\},\, \{2,\, 4\},\, \{3,\, 5\}\}.$$

Definition 11.9. A graph G for which $|V| = n$ and $E = \binom{V}{2}$ is called a *complete graph* on n vertices, and is denoted K_n. See Figure 11.2 for drawings of K_n for $n = 1,\, \dots,\, 5$.

Definition 11.10. Let G be a graph and k a positive integer. An *edge k-coloring* of G is an assignment to each edge of G of one of a collection of k different colors.

Figure 11.3 illustrates an edge 2-coloring of K_5, using colors red and blue, in which precisely those edges in Example 11.8 are colored red.

Definition 11.11. Let $G = (V,\, E)$ and $H = (V',\, E')$ be graphs. We say that H is a *subgraph* of G if

(i) $V' \subseteq V$,

(ii) $E' \subseteq E \cap \binom{V'}{2}$, in other words, every edge of H is also an edge in G and goes between two vertices of the subset V' of V.

We can now recast the definition of Ramsey numbers in terms of graphs:

Graph version of Definition 10.5. Let $k,\, l \in \mathbb{N}_{\geq 2}$. The Ramsey number $R(k,\, l)$ is the smallest $n \in \mathbb{N}$ such that, in any edge 2-coloring of K_n with the colors red and blue, there must exist either an entirely red K_k subgraph or an entirely blue K_l subgraph.

Remark 11.12. In Lecture 19, while discussing a completely different problem, we will define the same term "edge k-coloring" in a more restrictive manner (Definition 19.4) than above. Whether or not the more restrictive definition is intended depends on the context. Definition 11.10 is the standard terminology in the context of Ramsey theory, but not otherwise.