Twelveth Lecture: 28/4

Proof of Theorem 11.2. Let $k \geq 3$ be given and let n be an integer satisfying (11.2). We must prove that there exists an edge 2-coloring of K_n in which no K_k subgraph is monochromatic. The idea for the proof is to employ a *probabilistic argument:* instead of trying to describe such a coloring explicitly, we instead color the edges independently and uniformly at random and prove that, with positive probability, no K_k subgraph will be monochromatic.

To be precise, for each of the $\binom{n}{2}$ edges of K_n, toss a fair coin and color the edge red if the coin shows heads, blue otherwise. Do this independently for all the edges. Hence, every edge is colored red (resp. blue) with probability $1/2$, independently of all other edges. There are a total of $\binom{n}{k}$ K_k subgraphs in K_n. Order them arbitrarily and, for each $i = 1, \ldots, \binom{n}{k}$, let A_i be the event that the i:th subgraph is monochromatic. Clearly, for each i,

$$A_i = A_{i,r} \sqcup A_{i,b} \quad (\sqcup \text{ denotes disjoint union/mutual exclusivity}),$$

where $A_{i,r}$ (resp. $A_{i,b}$) is the event that the i:th subgraph is entirely red (resp. blue). Thus, by (11.4),

$$\mathbb{P}(A_i) = \mathbb{P}(A_{i,r}) + \mathbb{P}(A_{i,b}).$$

Consider $A_{i,r}$. We are dealing here with a fixed K_k subgraph. This graph has $\binom{k}{2}$ edges, order them arbitrarily. Let $A_{i,r,j}$ be the event that the j:th edge of this graph is colored red, for $j = 1, \ldots, \binom{k}{2}$. Then

$$A_{i,r} = \bigwedge_{j=1}^{\binom{k}{2}} A_{i,r,j}.$$

But $\mathbb{P}(A_{i,r,j}) = \frac{1}{2}$ for each j, and the events $A_{i,r,j}$ are independent for different j's. Hence, by (11.9),

$$\mathbb{P}(A_{i,r}) = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}}.$$

Clearly, $\mathbb{P}(A_{i,b}) = \mathbb{P}(A_{i,r})$ and hence

$$\mathbb{P}(A_i) = 2 \times 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}.$$

Let X be the event that *some* K_k subgraph is monochromatic. Then

$$X = \bigvee_{i=1}^{\binom{n}{k}} A_i$$

and so, by the Union Bound (11.7),

$$\mathbb{P}(X) \leq \sum_{i=1}^{\binom{n}{k}} \mathbb{P}(A_i) = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

By assumption, (11.2) holds, thus $\mathbb{P}(X) < 1$, equivalently $\mathbb{P}(\neg X) > 0$. In other words, when we color the edges of K_n independently and uniformly at random, then there is a strictly positive probability of obtainining no monochromatic K_k subgraph.

But what does this mean ? Well, there are $2^{\binom{n}{2}}$ possible edge 2-colorings of K_n. When we color the edges independently and uniformly at random, each of these $2^{\binom{n}{2}}$ possible colorings arises with equal probability, in other words any particular coloring arises with probability $2^{-\binom{n}{2}}$. Hence, if there are a total of t colorings which give rise to no monochromatic K_k subgraphs, then

$$\mathbb{P}(\neg X) = t \cdot 2^{-\binom{n}{2}}.$$

So $\mathbb{P}(\neg X) > 0 \Leftrightarrow t > 0 \Leftrightarrow t \geq 1$, since t is an integer. So we've proven that there is at least one edge 2-coloring of K_n which yields no monochromatic K_k subgraphs, v.s.v. $\square$

Remark 12.1. From the proof of Corollary 11.3 we can see that, as k grows, if $n = \lfloor 2^{k/2} \rfloor$ then the expression $\binom{n}{k} 2^{1-\binom{k}{2}}$ goes to zero very rapidly. Hence, the proof of Theorem 11.2 given above actually implies that, for reasonably large k and $n = \lfloor 2^{k/2} \rfloor$, a random edge 2-coloring of K_n will "work", i.e.: will yield no monochromatic K_k subgraphs, with extremely high probability. Hence, as a practical issue it is very easy to produce a coloring that works, simply by trial and error. However, it remains a (probably impossible !) open problem to describe explicitly *any* coloring that works.

Part 2: Graph Theory

The basic defintion of *graph* was already given in Definition 11.7. We'll get started in earnest with this topic next day, but let me finish today by just noting some terminology associated with standard variations on the basic definition:

Terminology 12.2. A *digraph* is a pair $G = (V, E)$, where E is now a set of *ordered* pairs of elements of V. One can think of a digraph as an ordinary graph but with an arrow on each edge, i.e.: each edge has a direction from one vertex to the other. We will see an example next day when we discuss the *Keycode Problem*.

In a *multigraph* $G = (V, E)$, the set E is allowed to contain repititions. In other words, one allows there to be multiple edges between the same pair of vertices. The *Bridges of Königsberg* problem, to be discussed next day, provides a typical example.

A *loop* is an edge from a vertex to itself. Hence if $G = (V, E)$ and we want to allow loops, then the set E should be allowed to contain pairs $\{v, v\}$. We will also encounter a graph with loops in the Keycode Problem.

In some textbooks, the word "graph" does not have the same meaning as in our Definition 11.7, but could incorporate one or more of the variations digraph, multigraph, graph with loops. Indeed, we will also reserve the right to abuse language in the coming lectures and trust the reader to understand from the context at hand what we mean by the term “graph”. In such cases, it is common to speak of what we termed a “graph” in Definition 11.7 as a *simple, loopless* graph.

Definition 12.3. Let $G = (V,\, E)$ be a simple, loopless graph with $|V| = n$ and suppose the vertices have been labelled $1,\, 2,\, \ldots,\, n$. The *adjacency matrix* of G is the $n \times n$ matrix $A = A_G = (a_{i,\,j})$ such that

$$a_{i,\,j} = \begin{cases} 1, & \text{if } \{i,\, j\} \in E, \\ 0, & \text{if } \{i,\, j\} \notin E. \end{cases}$$

The definition can be extended to the various variations discussed above, noting that:

(i) In a multigraph, $a_{i,\,j}$ would denote the number of edges between vertices i and j.
(ii) In a digraph, the adjacency matrix need not be symmetric, i.e.: $a_{i,\,j} \neq a_{j,\,i}$ is possible.
(ii) If $a_{i,\,i} > 0$ it means there is at least one loop at vertex i.