Thirteenth Lecture: 2/5

Definition 13.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. We say that G_1 and G_2 are *isomorphic*, and write $G_1 \cong G_2$, if there is a map $\phi : V_1 \to V_2$ such that
(i) ϕ is a bijection
(ii) $\{v_1, w_1\} \in E_1$ if and only if $\{\phi(v_1), \phi(w_1)\} \in E_2$.

If I were to draw two graphs on the blackboard "at random", then it is very likely that they would not be isomorphic *and* that you could easily find a reason why. Namely, there are lots of so-called *graph invariants* - that is, numbers associated to a graph which can be computed, and which must match for isomorphic graphs. Examples of such invariants are number of vertices, number of edges, number of connected components, degree sequence, girth, chromatic number etc etc. Note that most of these we haven't defined yet (though we will in due course) and some are in general much easier to compute than others, though if you just want to show two graphs are not isomorphic you just need to show that they have different values for some invariant, not compute the invariants exactly (in other words, if the two numbers in question differ by a lot then this might be easy to spot).

However, when graphs get large and pictorial representations get messy, with lots of tangled, criss-crossing edges, deciding whether or not two graphs are isomorphic needs to be approached with much more circumspection. The so-called *graph isomorphism problem* can be stated in a weaker or a stronger form, something typical for algorithmic problems in this area of math:

The Decision Problem. One seeks an algorithm, which is as efficient and general as possible, for deciding whether or not two graphs, given as pairs (V_1, E_1), (V_2, E_2), are isomorphic.

The Search Problem. This is an extension of the decision problem. One seeks an algorithm, as efficient and general as possible, which for two given inputs (V_1, E_1) and (V_2, E_2), both decides if the graphs are isomorphic and, if they are, produces an explicit isomorphism, i.e.: an explicit map $\phi : V_1 \to V_2$ satisfying the requirements of Definition 13.1.

The graph isomorphism problem seems to be hard, though how hard is not yet clear - it is still the subject of ongoing research and there have been significant developments even in recent years. See, for example, the following paper if you're curious: `http://arxiv.org/abs/1512.03547`

Example 13.2. The three graphs in Figure 13.1 on the homepage are all isomorphic, indeed the numberings shown in the figure describe appropriate isomorphisms. Note that the three drawings all look considerably different, so unless I told you they were all isomorphic this fact would probably have escaped your notice if you'd just glanced casually at the three drawings. By the way, this graph is known as *Petersen's graph.*

The top-left rendering is the most usual one.

Example 13.2 shows that the same graph can be drawn in many different ways and various pictorial representations of it could look very different from one another. One particular type of representation which seems natural to look for, from an aesthetic viewpoint if nothing else, is one in which no two edges cross one another, so no tangles. This property has a name:

Definition 13.3. A graph G is said to be *planar* if it can be drawn on a plane surface so that no two edges cross. Any such drawing is called a *plane graph*.

Example 13.4. The usual representations of K_n, $1 \leq n \leq 5$, are given in Figure 13.2 on the homepage. For $n \leq 3$ they are already plane. But K_4 is also planar since one of the diagonals can be moved outside the square (see Figure). It turns out, however, that K_5 cannot be untangled and is not planar (see Homework 2). Hence, K_n is not planar for any $n > 5$ either, because K_n contains copies of K_m whenever $n > m$. So K_n is planar if and only if $n \leq 4$.

Definition 13.5. A graph $G = (V,\, E)$ is said to be *bipartite* ifthere exist subsets $V_1,\, V_2$ of V such that
(i) $V_1 \neq \phi$, $V_2 \neq \phi$
(ii) $V_1 \cap V_2 = \phi$
(iii) $V = V_1 \cup V_2$
(iv) if $\{v,\, w\} \in E$, then one of v and w is in V_1 and the other is in V_2.
It is normal to write $G = (V_1,\, V_2,\, E)$ for a bipartite graph. Pictorially, a bipartite graph has two "sides" and every edge crosses from one side to the other.

Example 13.6. Let $m,\, n \in \mathbb{N}$. The *complete bipartite graph* $K_{m,n}$ is the unique bipartite graph $G = (V_1,\, V_2,\, E)$, up to isomorphism, for which $|V_1| = m$, $|V_2| = n$, $\{v_1,\, v_2\} \in E$ for all $v_1 \in V_1$ and $v_2 \in V_2$. Note that $K_{m,n}$ has a total of mn edges.

The graphs $K_{1,n}$, $K_{2,n}$ and $K_{3,3}$ are drawn in Figure 13.3 on the homepage. The drawing for $K_{1,n}$ is plane. That for $K_{2,n}$ isn't, but we can move vertex 2 to the right and get a plane drawing (see Figure), so $K_{2,n}$ is also planar. However, it turns out that $K_{3,3}$ is not planar (see Homework 2) and hence that $K_{m,n}$ is planar if and only if $\min\{m,\, n\} < 3$.

Non-planarity of K_5 and $K_{3,3}$ can be deduced from the fundamental result on plane graphs proven by Euler.

Theorem 13.7. *Let $G = (V,\, E)$ be a connected plane graph. Then*

$$v - e + r = 1, \tag{13.1}$$

where $v = |V|$ is the number of vertices, $e = |E|$ is the number of edges and r is the number of minimal enclosed regions.

Example 13.8. For the plane graph in Figure 13.4 on the homepage one has $v = 23$, $e = 33$ and $r = 11$, the regions being numbered as in the Figure. Hence $v - e + r = 1$, as the theorem says.

Proof of Theorem 13.7. The easiest way to prove the theorem is by induction on the number of edges.

BASE CASE: If G has one edge, then it must be a K_2, hence $v = 2$, $e = 1$ and $r = 0$, so yes, $v - e + r = 1$ in this case.

INDUCTION STEP: Suppose the theorem holds for all connected, plane graphs on n edges and let G be a connected, plane graph on $n + 1$ edges. We can certainly draw G one edge at a time, in such a way that it is always plane and connected and such that every edge is drawn on the outside of the previous figure. Let G' represent the drawing when one edge remains to be added. By the induction assumption, $v' - e' + r' = 1$, where the primes represent the various quantities in G' and $e' = e - 1 = n$.

When we now add the last edge, since G is connected two possibilities arise:

Case 1: This last edge joins two existing vertices. Thus no new vertex is created at this last step and $v = v'$. By joining two existing vertices we will create a new minimal enclosed region. However, we must create exactly one new such region, since G is plane so the new edge does not cross any existing edge. Hence $r = r' + 1$. So $v - e + r = v' - (e' + 1) + (r' + 1) = v' - e' + r' = 1$, v.s.v.

Case 2: This last edge joins an existing vertex to a new vertex. Thus one new vertex is created at this last step and $v = v' + 1$. The new vertex cannot subdivide any existing edge, as otherwise we'd create two new edges at this last step, not one. Nor can the last edge create a new enclosed region, as this would have to involve it crossing an existing edge. Hence $r = r'$ in this case. Thus, $v - e + r = (v' + 1) - (e' + 1) + r' = v' - e' + r' = 1$, v.s.v. □

Before leaving the subject of planarity, I wish to state the fundamental result about which graphs are planar. To do so, I need another definition:

Definition 13.9. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. We say that G' is a *one-step subdivision* of G if[1]

(i) $V' = V \sqcup \{x\}$, for some single vertex x

(ii) there is an edge $\{v, w\} \in E$ such that $E' = [E \cup \{\{v, x\}, \{x, w\}\}] \backslash \{\{v, w\}\}$.

In words, G' is gotten from G by inserting an extra vertex along one of its edges and thus dividing that edge into two.

More generally, we say that G' is a *subdivision* of G if G' can be obtained from G by a finite sequence of one-step subdivisions.

Theorem 13.10. (Kuratowski's Theorem) *A graph G is planar if and only if it contains no subgraph which is a subdivision of K_5 or $K_{3,3}$.*

[1] $\sqcup$ denotes disjoint union.

Note that in Examples 13.4 and 13.6 we have already discussed the "easy half" of this theorem, namely to show that neither K_5 nor $K_{3,3}$ is planar and hence that G is not planar if it possesses a subgraph which is a subdivision of either of them. The much harder part is to prove that these are basically the only two patterns which prevent a graph being planar. The proof of Kuratowski's Theorem is well beyond the scope of this course, so the interested reader may consult the literature.

Degrees. We've already touched on this terminology, but it's time for a formal definition:

Definition 13.11. Let $G = (V, E)$ be a graph and $v \in V$. The *degree* of V is the number of edges in G containing v, i.e.:

$$\deg(v) = \#\{w \in V : \{v, w\} \in E\}. \tag{13.2}$$

Another common notation for the degree is d_v. Note that the definition can be extended to multigraphs if we count every edge with its multiplicity.

Theorem 13.12. (Degree equation) *Let $G = (V, E)$ be a (multi)graph. Then*

$$\sum_{v \in V} deg(v) = 2|E|. \tag{13.3}$$

In particular, in any graph there must be an even number of vertices of odd degree.

Proof. Count all pairs (v, e), where $v \in V$ and e is an edge containing v. By definition, there are $\deg(v)$ pairs containing a given vertex v. Hence the total number of pairs is $\sum_{v \in V} \deg(v)$, the LHS of (13.3). On the other hand, each edge $e \in E$ contains precisely two vertices, in other words each edge $e = \{v, w\}$ gives rise to precisely two pairs, (v, e) and (w, e). Hence the total number of pairs must be twice the number of edges, i.e.: $2|E|$, v.s.v. Note that this proof works just as well for multigraphs. □

Example 13.13. (The NFL Problem) This is apparently a true story[2]. NFL stands for National Football League, the organisation which runs professional American football. At some point in the 1960s the league consisted of two so-called conferences, AFC and NFC. Each conference consisted of 13 teams. The league bosses wanted to make a schedule for the upcoming season in which

Each team would play 14 *games, of which* 9 *would be against teams in its own conference and* 5 *against teams in the other conference.*

It turns out (and it was a big fiasco at the time !) that such a schedule is impossible to implement. The requirement that each team play 5 matches against teams in the

[2] In fact, in the original story, the numbers were $14 - 11 - 3$ rather than $14 - 9 - 5$. The same principle applies, since what makes the schedule impossible is the odd number of games to be played by each team within its conference. However, when this number is 11 instead of 9, there is another way to see it's impossible, at least if we assume that no two teams meet more than once. In that case, each team plays 11 of the other 12, hence the teams can be paired off into pairs which do not meet. But such a pairing requires an even number of teams in total. Note, however, that the solution based on the degree equation is superior in that it does not assume that teams meet at most once.

other conference *can* be satisfied - I'll leave it as an exercise for you to make such a schedule. However, it is impossible to satisfy the requirement that each team play 9 matches against teams in its own conference.

For take one of the conferences, say AFC, and suppose a schedule meeting our demands existed. Now consider the graph $G = (V, E)$ in which V is the set of 13 AFC teams and E the set of matches within AFC - in other words, $\{v, w\} \in E$ if and only if the schedule includes a match between teams v and w. In fact, we can allow the possibility that a pair of teams meet more than once (say home and away), in which case G becomes a multigraph with the multiplicity of an edge being equal to the number of times the corresponding teams meet. Our demands would imply that every vertex in G had degree 9. Since there are 13 vertices, we'd thus have a graph in which an odd number of vertices have odd degree, contradicting Theorem 13.12.