Nineteenth Lecture: 18/5

The augmenting path algorithm can in fact be employed in an arbitrary graph, not just a bipartite graph. This is because of the following observation:

Proposition 19.1. *Let $G = (V, E)$ be any graph and M a matching in G. If M is not a maximum matching, then there exists an M-augmenting path.*

Proof. Let M^* be a maximum matching and let H be the subgraph of G consisting of those edges which occur in precisely one of M and M^*, but not both. Since M and M^* are both matchings, the degree of any vertex in H is at most 2. Hence, H has no branching points and each of its connected components must be either a simple cycle or a simple path. Moreover, in any such component, edges must alternate between M and M^*. This implies that any cycle must have even length and include an equal number of edges from M and M^*. But $|M^*| > |M|$, so there must be at least one component of H which is a path of odd length, in which edges alternate between M and M^* and the first edge is in the latter. Moreover, the first and last vertices along the path are unmatched in M, as otherwise this path could be extended to a larger connected subgraph of H. Hence the path is an M-augmenting path, v.s.v. $\square$

Hall's theorem can also be generalised using the following concept:

Definition 19.2. Let $G = (X, Y, E)$ be a bipartite graph. For a subset $A \subseteq X$, the *deficiency* of A, denoted ∂_A, is defined as $\partial_A := \max\{0, |A| - |N(A)|\}$. The deficiency of X, denoted d_X, is defined as $d_X := \max_{A \subseteq X} \partial_A$.

Theorem 19.3. (Extended Hall's theorem) *Let $G = (X, Y, E)$ be a bipartite graph. Then the maximum size of a matching in G is $|X| - d_X$.*

Proof. If $d_X = 0$ then this is just Hall's theorem, so we may suppose $d_X > 0$.

Let M be any matching in G. There exists a subset $A \subseteq X$ such that $|N(A)| = |A| - d_X$. Now vertices in A can only be matched a priori with vertices in $N(A)$. Hence, at least d_X of the vertices in A must be left unmatched, and thus a fortiori at least d_X of the vertices in X must be left unmatched. This proves that $|M| \leq |X| - d_X$ for any matching.

Conversely, let $G = (X, Y, E)$ be a bipartite graph with deficiency $d_X > 0$. Let $G' = (X, Y', E')$ be the graph gotten by adding d_X new vertices to Y to form Y', and then inserting an edge from every vertex of X to every vertex of $Y' \backslash Y$. Since we in this way increase the number of neighbors of every vertex in X by d_X, the graph G' has zero deficiency. Hence, by Hall's theorem, it possesses a perfect matching for X. But in this matching, no more than d_X vertices can be matched with vertices in $Y' \backslash Y$, since there are only d_X vertices in the latter, by construction. Hence, at least $|X| - d_X$ vertices of X are matched with vertices in Y. These pairs constitute a matching in the original graph G, so G has a matching of size at least $|X| - d_X$. This completes the proof. $\square$

Definition 19.4. Let $G = (V, E)$ be a graph and k a positive integer. An *edge k-coloring* of G is a function $f : E \to \{1, 2, \ldots, k\}$ such that $f(e_1) \neq f(e_2)$ whenever the edges e_1 and e_2 have a common vertex.

The *edge chromatic number* of G, denoted $\Phi(G)$, is the smallest k for which there exists an edge k-coloring of G.

The vertex- and edge-coloring problems may seem similar, but in some sense the latter is in fact much simpler. First note the following simple analogue of Proposition 16.6:

Proposition 19.5. *For any graph G one has*

$$\Phi(G) \geq \Delta(G). \tag{19.1}$$

Proof. Obvious. All the edges incident to a given vertex must be assigned different colors in any edge coloring. □

It is already striking that the obvious lower bound for $\Phi(G)$ is basically the same as the (most) obvious upper bound for $\chi(G)$, as proven in Theorem 16.13. But, in fact, the relationship between $\Phi(G)$ and $\Delta(G)$ is much closer. The key to this is the following:

Theorem 19.6. *Let $G = (X, Y, E)$ be a bipartite graph. Then $\Phi(G) = \Delta(G)$.*

Proof. The central idea in the proof is very similar to the augmenting path idea for constructing maximum matchings. Formally, one proceeds by induction on the number of edges in G. If G contains a single edge then obviously $\Phi(G) = \Delta(G) = 1$. So suppose the theorem holds for any bipartite graph with at most $m \geq 1$ edges and let $G = (X, Y, E)$ be a bipartite graph with $m + 1$ edges. Pick any edge, call it $e_{00} = \{x_0, y_0\}$. Removal of e_{00} leaves a bipartite graph G' with m edges.

CASE 1: $\Delta(G') = \Delta(G) - 1$. By induction, G' can be edge-colored with $\Delta(G) - 1$ colors. Then the $\Delta(G)$:th color can be used on e_{00}.

CASE 2: $\Delta(G') = \Delta(G)$. We can still apply the induction hypothesis to conclude that G' has an edge coloring with $\Delta(G)$ colors. Fix such a coloring $\mathcal{C}$ - in particular, fix a list of $\Delta(G)$ available colors. In the graph G' the vertices x_0 and y_0 each have degree at most $\Delta(G) - 1$. Hence there must be some color α on the list which has not been used in $\mathcal{C}$ on an edge incident to x_0. Similarly, there must be some color β on the list not used in $\mathcal{C}$ on an edge incident to y_0. If $\alpha = \beta$, then we can use this color on e_{00} and obtain an edge $\Delta(G)$-coloring of G.

So we may suppose that $\alpha \neq \beta$. In other words, we may assume that there *is* some edge incident to y_0 on which the color α has been used. Call this edge $e_{01} = \{x_1, y_0\}$. What we would like to do now is to swap α for β on e_{01} and then use α on e_{00}. The only thing that can disallow this is if there is an edge incident to x_1 on which β has been used. So we may assume there is such an edge, call it $e_{11} = \{x_1, y_1\}$.

Keep iterating this argument. We can stop and perform the $\alpha \leftrightarrow \beta$ swap as soon as we encounter either an X-vertex where the color β is free or a Y-vertex where the color α is free. But such a vertex *must* eventually be encountered since the graph is finite and if the sequence $x_0, y_0, x_1, y_1, x_2, \ldots$ of vertices encountered a repetition, then we would have two edges in G' with the same color and a vertex in common. □

Remark 19.7. In fact, the lower bound in (19.1) is almost sharp for any graph whatsoever. *Vizing's theorem* states that $\Phi(G) \leq \Delta(G) + 1$, for *any* graph !! The proof is similar in spirit to that of Theorem 19.6 and not too difficult, though we do not have time for it. See, if you're interested, `https://en.wikipedia.org/wiki/Vizing's_theorem`