Fourth Lecture: 11/4

Definition 4.1. A sequence $(a_n)_{n=1}^{\infty}$ of complex numbers is said to be *defined recursively*, or to *satsify a recurrence relation*, if there exists a fixed $k \in \mathbb{N}$, called the *degree* of the recurrence, and a function $f : \mathbb{C}^k \to \mathbb{C}$ such that, for all $n \in \mathbb{N}$,

$$a_{n+k} = f(a_{n+k-1}, a_{n+k-2}, \dots, a_n). \tag{4.1}$$

Observe that a sequence satisfying (4.1) is completely determined by the values $a_1, a_2, \dots, a_k$.

Remark 4.2. It will sometimes be convenient to index sequences from a different starting point than $n = 1$, for example it is often useful to start from $n = 0$, especially in situations where the index refers to "time". We will adjust the starting index in the text below as the situation demands without further comment.

Two basic issues arise in the study of recursively defined sequences:

The Combinatorial Problem. The function f may not be given, and the first step is to find it. This usually involves some kind of "combinatorial reasoning".

The Algebraic Problem. If possible, solve the recurrence (4.1) to find an "explicit formula" for a_n as a function of n. This is a special case of the *computational problem* of computing the elements of the sequence as efficiently as possible. The recursion itself leads to relatively efficient computation - just write a program with a loop. But by an "explicit formula" we have in mind something even better. Moreover, for many applications it may suffice to be able to make a good estimate of a_n for large n, and an explicit formula may give this insight directly, requiring only minimal computation.

Here is a very simple example to illustrate:

Example 4.3. Let a_n denote the number of subsets of an n-element set, say $\{1, 2, \dots, n\}$ WLOG. Clearly $a_0 = 1$, since only the empty subset occurs. Moreover, $a_{n+1} = 2a_n$. For among the subsets of $\{1, 2, \dots, n+1\}$ we can distinguish two types: those which contain $n+1$ and those which don't. Those of each type are in 1-1 correspondence with subsets of $\{1, 2, \dots, n\}$, hence the recursion.

So the sequence $(a_n)_{n=0}^{\infty}$ is defined recursviely by

$$a_0 = 1, \quad a_{n+1} = f(a_n), \text{ where } f(x) = 2x. \tag{4.2}$$

Here the algebraic problem is very simple, we can immediately see that $a_n = 2^n$. Note that this explicit formula immediately tells us the order of magnitude of a_n for any particular n, namely $2^n = 10^{n \log_{10} 2} \approx 10^{0.3\,n}$.

Just as, in general, very few algebraic equations can be solved exactly, there are very few functions f for which (4.1) can be solved to yield an explicit formula for a_n. Fortunately, those for which an exact solution procedure exists are also those which tend to arise most commonly. We will in this course focus mostly[1] on *linear* recurrences.

Definition 4.4. The sequence $(a_n)_{n=1}^{\infty}$ is said to satisfy a *homogeneous linear recurrence (HLR)* if the function f in (4.1) is homogeneous and linear, by which we mean that there exist constants $c_1, c_2, \dots, c_k$ such that $f(x_1, x_2, \dots, x_k) = \sum_{i=1}^{k} c_i x_i$.

Homogeneous linear recurrences can always be solved explicitly. We will leave the formal statement of this result and its proof to the next lecture. For the remainder of this lecture, we will concentrate on working through some examples to build intuition. Note that Example 4.3 was already a simple example of a HLR.

Example 4.5 (Fibonacci numbers). Fibonnaci was apparently interested in studying the reproductive behaviour of rabbits (who are a good choice of species to study for this purpose because their rate of reproduction is unusually fast for mammals). Fibonacci's model makes the following assumptions. Some of them obviously sound stupid on the level of individual rabbits, but in such cases one should instead imagine that they are statements about average behaviour in a large population - see the remark below.

ASSUMPTION 1: Rabbits live forever.
ASSUMPTION 2: Rabbits form monogamous pairs.
ASSUMPTION 3: Rabbit pregnancy lasts one month.
ASSUMPTION 4: Rabbit childhood lasts one month.
ASSUMPTION 5: Adult females conceive new offspring as soon as the last batch have been born.
ASSUMPTION 6: Each conception results in a pair of twins, one of each sex.

Remark: Assumptions 3-6 should be thought of as representing average behaviour, in which case Assumption 2 becomes superfluous. Assumption 1 is, however, a serious restriction and obviously implies that the model is unrealistic over longer time periods. Hoever, one can still consider it as a first step in trying to understand how quickly a population of rabbits will proliferate over shorter time scales.

Suppose we begin with a single pair of newborn rabbits. For each $n \geq 1$, let f_n denote the number of rabbit pairs living after $n-1$ months[2]. Thus $f_1 = 1$ by assumption. Also, $f_2 = 1$ since after one month the newborn pair will have grown up but not yet produced any offspring. Now I claim that, for all $n \geq 1$,

$$f_{n+2} = f_{n+1} + f_n. \tag{4.3}$$

To see this, first write $f_n = v_n + b_n$, where v_n is the number of adult pairs after $n-1$ months and b_n is the number of newborn pairs at this time. Then observe that

(i) $v_{n+2} = f_{n+1}$ since every rabbit which was alive one month previously will now be an adult,

[1]though not entirely, see the discussion of Catalan numbers in Lecture 6.

[2]Obviously, it would seem more natural to define f_n to be the number of rabbit paris alive after n months, i.e.: to start from time $n = 0$ and have as initial conditions $f_0 = f_1 = 1$. I choose to start from $n = 1$ because that is what Biggs' book does.

(ii) $b_{n+2} = f_n$ since only those rabbit pairs which were alive two months previously will have produced offspring in the previous month, since they first needed to become adults the month before.
Then (4.3) follows from (i) and (ii). To summarise, the sequence $(f_n)_{n=1}^{\infty}$ of Fibonacci numbers is defined by the recurrence

$$f_1 = 1, \quad f_2 = 1, \quad f_{n+2} = f_{n+1} + f_n \;\; \forall\, n \geq 1. \tag{4.4}$$

Here, in the notation of (4.1), $f(x, y) = x + y$, so (4.4) is a HLR.

It's quite easy to understand intuitively the method for solving (4.4), but let me for the sake of completeness (and as preparation for the formal theorems to come in Lecture 5) be a bit more formal. Let

$$l^{\infty} := \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{C}\}. \tag{4.5}$$

In words, l^{∞} is the vector space of all infinite sequences of complex numbers[3]. It is obviously an infinite-dimensional vector space (in fact, the dimension is uncountable - can you prove this ?). Now let

$$V = \{(x_n)_{n=1}^{\infty} \in l^{\infty} : x_{n+2} = x_{n+1} + x_n \; \forall\, n \geq 1\}. \tag{4.6}$$

Firstly, I claim that V is a subspace of l^{∞} - indeed this is equivalent to the fact that the recurrence is a HLR. The proof is simple, but let me give it for the sake of completeness. To show that V is a subspace, we must show that it is closed under vector addition and scalar multiplication.

Closure under addition: Let $\boldsymbol{x} = (x_n)$ and $\boldsymbol{y} = (y_n)$ be elements of V, thus $x_{n+2} = x_{n+1} + x_n$ and $y_{n+2} = y_{n+1} + y_n$. Let $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. Then

$$z_{n+2} \overset{\text{def}}{=} x_{n+2} + y_{n+2} = (x_{n+1} + x_n) + (y_{n+1} + y_n) =$$
$$= (x_{n+1} + y_{n+1}) + (x_n + y_n) \overset{\text{def}}{=} z_{n+1} + z_n, \quad \text{v.s.v.}$$

Closure under scalar multiplication: Let $\boldsymbol{x} = (x_n) \in V$ and $\alpha \in \mathbb{C}$. Let $\boldsymbol{z} := \alpha \boldsymbol{x}$. Then

$$z_{n+2} \overset{\text{def}}{=} \alpha x_{n+2} = \alpha(x_{n+1} + x_n) = (\alpha x_{n+1}) + (\alpha x_n) \overset{\text{def}}{=} z_{n+1} + z_n, \quad \text{v.s.v.}$$

Next, I claim that $\dim(V) = 2$. Intuitively, the reason is that a sequence in V is completely determined by its first two terms. More precisely, there is a vector-space isomorphism $\phi : \mathbb{C}^2 \to V$ given by

$$\phi((a, b)) = \text{the unique sequence } (x_n) \in V \text{ such that } x_1 = a, \, x_2 = b.$$

Hence, solving (4.4) reduces to determining a basis for V. At this point one needs to make an "inspired guess", and the right guess is to set $x_n = \alpha^n$ and solve for $\alpha \in \mathbb{C}$. Thus,

$$x_{n+2} = x_{n+1} + x_n \Leftrightarrow \alpha^{n+2} = \alpha^{n+1} + \alpha^n.$$

[3] The notation is standard in functional analysis.

Now $\alpha \neq 0$ as otherwise the sequence (x_n) would be identically zero and hence not a basis vector. Thus we can cancel α^n from the equation and get an equation which is independent of n:

$$\alpha^2 = \alpha + 1.$$

This is called the *auxiliary equation* for the recurrence (4.4). It is a quadratic equation with two roots[4],

$$\alpha_1 = \gamma = \frac{1+\sqrt{5}}{2}, \quad \alpha_2 = \frac{-1}{\gamma} = \frac{1-\sqrt{5}}{2}.$$

Hence a basis for V is given by the two sequences $(\gamma^n)_{n=1}^{\infty}$ and $((-1/\gamma)^n)_{n=1}^{\infty}$. Thus the Fibonacci sequence $(f_n)_{n=1}^{\infty}$ must be a linear combination of these two, i.e.: there exist constants C_1, C_2 such that

$$f_n = C_1 \cdot \gamma^n + C_2 \cdot \left(\frac{-1}{\gamma}\right)^n.$$

To determine C_1 and C_2 we insert the initial conditions $n = 1$ and $n = 2$:

$$n = 1: \quad f_1 = 1 = C_1 \cdot \gamma + C_2 \cdot \left(\frac{-1}{\gamma}\right),$$

$$n = 2: \quad f_2 = 1 = C_2 \cdot \gamma^2 + C_2 \cdot \frac{1}{\gamma^2}.$$

This is just a system of two linear equations in two unknowns, so a standard Gauss elimination problem, though with somewhat ugly coefficients. You can check that the solution is $C_1 = \frac{1}{\sqrt{5}}$, $C_2 = \frac{-1}{\sqrt{5}}$ and hence that the explicit solution to (4.4) is

$$f_n = \frac{1}{\sqrt{5}}\left(\gamma^n + (-1)^{n+1}\gamma^{-n}\right). \tag{4.7}$$

Given a HLR of degree k, the corresponding vector space V will be k-dimensional. The auxiliary equation will be a polynomial equation of degree k so, if we're lucky, it will have k distinct roots in the complex numbers[5], which will give us a complete basis of k vectors for V. The only thing that can possibly go wrong is that the auxiliary equation has one or more repeated roots. However, this situation can be handled. The complete proof will be given in Lecture 5, for the moment we just do an example:

Example 4.6. Let's solve the recurrence

$$u_0 = 1, \quad u_1 = 2, \quad u_{n+2} = 6u_{n+1} - 9u_n \;\; \forall\, n \geq 0.$$

The auxiliary equation is $\alpha^2 - 6\alpha + 9 = 0$, which has a repeated root $\alpha_{1,2} = 3$. Hence the sequence (3^n) is one basis vector. It turns out that the second one is given by the sequence $(n \cdot 3^n)$. Hence, there exist constants C_1, C_2 such that

$$u_n = (C_1 + C_2 n)3^n.$$

[4] γ is standard notation for the golden ratio.

[5] Recall the *Fundamental Theorem of Algebra*, which states that every polynomial with complex coefficients can be completely factorised in $\mathbb{C}$ - the technical terminology being that $\mathbb{C}$ is an algebraically closed field.

We insert the initial conditions

$$n = 0: \quad u_0 = 1 = C_1, \quad n = 1: \quad 2 = u_1 = 3(C_1 + C_2) \Rightarrow C_2 = -1/3.$$

Hence, $u_n = \left(1 - \frac{n}{3}\right) 3^n$.

Inhomogeneous linear recurrences (ILRs). We are now concerned with the situation where the function f in (4.1) is allowed to depend on n, but in a very specific way. Namely, we assume that there are constants $c_1, c_2, \ldots, c_k$ as before, and a sequence $(b_n)_{n=1}^{\infty}$ such that

$$a_{n+k} = \sum_{i=1}^{k} c_i a_{n+k-i} + b_n. \tag{4.8}$$

Note that this seems to be a completely trivial statement, since one could set $c_1 = \cdots = c_k = 0$ and just be left with the statement that $a_n = b_n$, i.e.: with a statement that tells you nothing at all. However, an equation like (4.8) will be useful if the sequence b_n is "given explicitly" (unlike a_n) and is of a fairly simple form. There are basically two kinds of sequences which the theory can easily handle: exponential functions $b_n = \beta^n$ and polynomials $b_n = p(n)$, where β is a fixed complex number and $p(x) \in \mathbb{C}[x]$ is a fixed polynomial.

Once again, we postpone general theorems to the next lecture and just do some examples today to build intuition:

Example 4.7. Solve the recurrence

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+2} = 6u_{n+1} - 8u_n + 3^n \ \forall\, n \geq 0. \tag{4.9}$$

In the notation of (4.8) we have $b_n = 3^n$, which is an exponential function with $\beta = 3$, so the theory says it is manageable. The idea for solving (4.9) should be familiar from linear algebra. Let V play the same role as before, i.e.:

$$V = \{(x_n) \in l^{\infty} : x_{n+2} = 6x_{n+1} - 8x_n \ \forall\, n \geq 0\}.$$

Let $W \subseteq l^{\infty}$ consist of those sequences which satisfy the inhomogeneous recurrence in (4.9), i.e.:

$$W = \{(x_n) \in l^{\infty} : x_{n+2} = 6x_{n+1} - 8x_n + 3^n \ \forall\, n \geq 0\}.$$

Then one can show (I leave it to yourselves to write out all the gruesome details of a proof !) that W is a coset of the subspace V, i.e.: if $\boldsymbol{u}_p$ is any element of W then $W = V + \boldsymbol{u}_p$. In other words, the general solution to (4.9) is

$$u_n = u_{h,n} + u_{p,n},$$

where $(u_{h,n})$ is the general solution to the corresponding homogeneous equation (i.e.: the same recurrence but without the 3^n term) and u_p is any *particular* solution to the inhomogeneous recurrence. The former is determined as before: the auxiliary equation is $\alpha^2 - 6\alpha + 8 = 0$, which as two roots $\alpha_1 = 4$, $\alpha_2 = 2$, thus $u_{h,n} = C_1 \cdot 4^n + C_2 \cdot 2^n$.

The reason why the recurrence (4.9) is manageable is because it is possible to write down a particular solution to it, namely it will have the form $u_{p,n} = C_3 \cdot 3^n$ for some constant C_3. We first solve for C_3 by insertion into the recurrence:

$$u_{p,\,n+2} = 6u_{p,\,n+1} - 8u_{p,\,n} + 3^n \Rightarrow C_3 \cdot 3^{n+2} = 6C_3 \cdot 3^{n+1} - 8C_3 \cdot 3^n + 3^n$$
$$(\text{cancel } 3^n) \Rightarrow 9C_3 = 18C_3 - 8C_3 + 1 \Rightarrow C_3 = -1.$$

Hence, the general solution to (4.9) is

$$u_n = C_1 \cdot 4^n + C_2 \cdot 2^n - 3^n.$$

Only now do we insert the initial conditions:

$$n = 0: \quad u_0 = 1 = C_1 + C_2 - 1 \Rightarrow C_1 + C_2 = 2,$$
$$n = 1: \quad u_1 = 1 = 4C_1 + 2C_2 - 3 \Rightarrow 4C_1 + 2C_2 = 4.$$

After the usual Gauss elimination we get $C_1 = 0$, $C_2 = 2$. Hence,

$$u_n = 2 \cdot 2^n - 3^n = 2^{n+1} - 3^n.$$

So what more can go wrong ? Well, what if one of the roots of the auxiliary equation coincides with β, where $b_n = \beta^n$ in (4.8) ? Once again, this can always be dealt with, as we will see formally in Lecture 5. For now, just another couple of examples:

Example 4.8. Solve

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+2} = 6u_{n+1} - 8u_n + 4^n \ \ \forall\, n \geq 0.$$

This time we can't take $u_{p,\,n} = C_3 \cdot 4^n$, because this is already subsumed by $u_{h,\,n}$. It turns out we can take $u_{p,\,n} = C_3 n \cdot 4^n$. Insertion gives

$$u_{p,\,n+2} = 6u_{p,\,n+1} - 8u_{p,\,n} + 4^n \Rightarrow C_3(n+2)4^{n+2} = 6C_3(n+1)4^{n+1} - 8C_3 n 4^n + 4^n$$
$$\Rightarrow C_3(n+2)4^{n+2} - 6C_3(n+1)4^{n+1} + 8C_3 n 4^n = 4^n$$
$$\Rightarrow C_3 n 4^n(4^2 - 6 \cdot 4 + 8) + C_3 4^n(2 \cdot 4^2 - 6 \cdot 4) = 4^n.$$

Now the crucial point is that the first bracketed number is zero, so that the term with $n4^n$ disappears from the LHS, as it must since no such term is on the RHS. The second bracketed number is non-zero, so now we can obtain a unique value for C_3 by cancelling 4^n, i.e.: $C_3 = 1/8$. Thus,

$$u_n = \left(C_1 + \frac{n}{8}\right) 4^n + C_2 \cdot 2^n.$$

We now insert the initial conditions:

$$n = 0: \quad u_0 = 1 = C_1 + C_2,$$
$$n = 1: \quad u_1 = 1 = 4\left(C_1 + \frac{1}{8}\right) + 2C_2 \Rightarrow 4C_1 + 2C_2 = \frac{1}{2}.$$

Gauss elimination yields $C_1 = -3/4$, $C_2 = 7/4$. Hence, finally,

$$u_n = \left(\frac{-3}{4} + \frac{n}{8}\right) 4^n + \frac{7}{4} \cdot 2^n.$$

Example 4.9. Solve

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+2} = 8u_{n+1} - 16u_n + 4^n \ \ \forall\, n \geq 0.$$

This time the auxiliary equation $\alpha^2 - 8\alpha + 16 = 0$ has a repeated root $\alpha_{1,2} = 4$ so $u_{h,n} = (C_1 + C_2 n)4^n$. To find a suitable $u_{p,n}$ we have to go "one more step up", namely $u_{p,n} = C_3 n^2 \cdot 4^n$. We first solve for C_3 by insertion:

$$\begin{aligned} u_{p,n+2} = 8u_{p,n+1} - 16u_{p,n} + 4^n \Rightarrow C_3(n+2)^2 4^{n+2} &= 8C_3(n+1)^2 4^{n+1} - 16C_3 n^2 4^n + 4^n \\ \Rightarrow C_3(n+2)^2 4^{n+2} - 8C_3(n+1)^2 4^{n+1} + 16C_3 n^2 4^n &= 4^n \\ \Rightarrow C_3 n^2 4^n(4^2 - 8\cdot 4 + 16) + C_3 n 4^n(2\cdot 2\cdot 4^2 - 8\cdot 2\cdot 4) + C_3 4^n(2^2\cdot 4^2 - 8\cdot 4) &= 4^n. \end{aligned}$$

This time, the coefficients of $n^2 4^n$ and $n4^n$ on the LHS are both zero, as desired, and we're left with $32C_3 = 1 \Rightarrow C_3 = 1/32$. Thus,

$$u_n = \left(C_1 + C_2 n + \frac{n^2}{32}\right)4^n.$$

We now insert the initial conditions:

$$\begin{aligned} n = 0: \quad u_0 = 1 = C_1, \\ n = 1: \quad u_1 = 1 = 4\left(C_1 + \frac{C_2}{8} + \frac{1}{32}\right) \Rightarrow \cdots \Rightarrow C_2 = -\frac{25}{32}. \end{aligned}$$

Hence, finally,

$$u_n = \left(1 - \frac{25n}{32} + \frac{n^2}{32}\right)4^n.$$

We didn't have time to do any examples today in which b_n is a polynomial in (4.8), but we'll do some next day. We finish off by noting that the examples we've done today seem to involve somewhat "ad hoc" guessing, especially for how to deal with "special cases" in which the auxiliary equation has repeated roots, or one or more of its roots coincide with the exponent in the inhomogeneous part of the recurrence. It turns out that there is, in fact, method to this madness, but to explain what's "really going on", we need to introduce so-called *generating functions*, which we will do next time.