Fifth Lecture: 13/4

If b_n in (4.8) is given by a polynomial of degree l, then one's guess for $u_{p,\,n}$ should be of the form

$$u_{p,\,n} = n^r \left(\sum_{i=0}^{l} \kappa_i n^i \right), \tag{5.1}$$

where r is the multiplicity of one as a root of the auxiliary equation and $\kappa_0, \ldots, \kappa_l$ are constants to be determined by substitution into the recurrence. We illustrate the method with the next three examples:

Example 5.1. Let's solve the recurrence

$$u_0 = 1, \quad u_1 = 3, \quad u_{n+2} = 6u_{n+1} - 8u_n + 2n \ \ \forall\, n \geq 0.$$

Here $b_n = 2n$, a polynomial of degree one, so $l = 1$ in (5.1). The auxiliary equation is $\alpha^2 - 6\alpha + 8 = 0$, which has roots $\alpha_1 = 2, \alpha_2 = 4$. Thus $r = 0$ in (5.1), since 1 is not a root of the auxiliary equation. We begin by determining $(u_{p,\,n})$ and, according to (5.1), our guess should be

$$u_{p,\,n} = C_3 + C_4 n.$$

Insertion into the recurrence gives

$$\begin{aligned} u_{p,\,n+2} = 6u_{p,\,n+1} - 8u_{p,\,n} + 2n \Leftrightarrow C_3 + C_4(n+2) &= 6[C_3 + C_4(n+1)] - 8[C_3 + C_4 n] + 2n \\ &\Rightarrow [(C_3 + 2C_4) - 6(C_3 + C_4) + 8C_3] + n[C_4 - 6C_4 + 8C_4] = 2n \\ &\Rightarrow 3C_3 - 4C_4 = 0 \ \text{ and } \ 3C_4 = 2 \\ &\Rightarrow C_3 = \frac{8}{9}, \quad C_4 = \frac{2}{3}. \end{aligned}$$

We have, as usual, $u_{h,\,n} = C_1 \cdot 2^n + C_2 \cdot 4^n$. Hence, the general solution to the recurrence is

$$u_n = C_1 \cdot 2^n + C_2 \cdot 4^n + \frac{8}{9} + \frac{2n}{3}.$$

We now insert the initial conditions

$$\begin{aligned} n = 0: \quad u_0 = 1 = C_1 + C_2 + \frac{8}{9} \Rightarrow C_1 + C_2 = \frac{1}{9}, \\ n = 1: \quad u_1 = 3 = 2C_1 + 4C_2 + \frac{8}{9} + \frac{2}{3} \Rightarrow 2C_1 + 4C_2 = \frac{13}{9}. \end{aligned}$$

Gauss elimination yields $C_1 = -1/2$, $C_2 = 11/18$. Hence, finally,

$$u_n = \frac{1}{18}\left(-9 \cdot 2^n + 11 \cdot 4^n + 16 + 12n\right).$$

Example 5.2. Let's solve the recurrence

$$u_0 = 1, \quad u_1 = 3, \quad u_{n+2} = 4u_{n+1} - 3u_n + 2n + 1 \ \ \forall\, n \geq 0.$$

Here $b_n = 2n + 1$ is of degree one as before, so $l = 1$ in (5.1). The auxiliary equation is $\alpha^2 - 4\alpha + 3 = 0$, which has roots $\alpha_1 = 1, \alpha_2 = 3$. Thus $r = 1$ in (5.1), since 1 is a simple root of the auxiliary equation. We begin by determining $(u_{p,n})$ and, according to (5.1), our guess should be

$$u_{p,n} = n(C_3 + C_4 n) = C_3 n + C_4 n^2.$$

Insertion into the recurrence gives

$$\begin{aligned} u_{p,n+2} &= 4u_{p,n+1} - 3u_{p,n} + 2n + 1 \\ \Leftrightarrow C_3(n+2) + C_4(n+2)^2 &= 4[C_3(n+1) + C_4(n+1)^2] - 3[C_3 n + C_4 n^2] + 2n + 1 \\ \Rightarrow [(2C_3 + 4C_4) - 4(C_3 + C_4)] &+ n[(C_3 + 4C_4) - 4(C_3 + 2C_4) + 3C_3] + \\ &+ n^2[C_4 - 4C_4 + 3C_4] = 2n + 1. \end{aligned}$$

Notice that the coefficient of n^2 on the LHS is identically zero. Hence, identifying the remaining coefficients on the left and right we get $-2C_3 = 1$ and $-4C_4 = 2$, so $C_3 = C_4 = -1/2$.

We have $u_{h,n} = C_1 \cdot 1^n + C_2 \cdot 3^n = C_1 + C_2 \cdot 3^n$, so the general solution to the recurrence is

$$u_n = C_1 + C_2 \cdot 3^n - \frac{n}{2}(1+n).$$

We now insert the initial conditions

$$\begin{aligned} n = 0: \quad u_0 = 1 &= C_1 + C_2, \\ n = 1: \quad u_1 = 3 &= C_1 + 3C_2 - \frac{1}{2}(1+1) \Rightarrow C_1 + 3C_2 = 4. \end{aligned}$$

Gauss elimination yields $C_1 = -1/2$, $C_2 = 3/2$. Hence, finally,

$$u_n = \frac{1}{2}\left[3^{n+1} - (1 + n + n^2)\right].$$

Example 5.3. Let's solve the recurrence

$$u_0 = 1, \quad u_1 = 3, \quad u_{n+2} = 2u_{n+1} - u_n + 2n + 1 \ \ \forall\, n \geq 0.$$

Here $b_n = 2n+1$ is of degree one as before, so $l = 1$ in (5.1). The auxiliary equation is $\alpha^2 - 2\alpha + 1 = 0$, which has the repeated root $\alpha_{1,2} = 1$. Thus $r = 2$ in (5.1). We begin by determining $(u_{p,n})$ and, according to (5.1), our guess should be

$$u_{p,n} = n^2(C_3 + C_4 n) = C_3 n^2 + C_4 n^3.$$

Insertion into the recurrence gives

$$\begin{aligned} u_{p,n+2} &= 2u_{p,n+1} - u_{p,n} + 2n + 1 \\ \Leftrightarrow C_3(n+2)^2 + C_4(n+2)^3 &= 2[C_3(n+1)^2 + C_4(n+1)^3] - [C_3 n^2 + C_4 n^3] + 2n + 1 \\ \Rightarrow [(4C_3 + 8C_4) - 2(C_3 + C_4)] &+ n[(4C_3 + 12C_4) - 2(2C_3 + 3C_4)] + \\ + n^2[(C_3 + 6C_4) - 2(C_3 + 3C_4) + C_3] &+ n^3[C_4 - 2C_4 + C_4] = 2n + 1. \end{aligned}$$

Notice that the coefficients of n^2 and n^3 on the LHS are both identically zero. Hence, identifying the remaining coefficients on the left and right we get $2C_3 + 6C_4 = 1$ and $6C_4 = 2$, so $C_3 = -1/2$ and $C_4 = 1/3$.

We have $u_{h,n} = (C_1 + C_2 n) \cdot 1^n = C_1 + C_2 n$, so the general solution to the recurrence is

$$u_n = C_1 + C_2 n - \frac{n^2}{2} + \frac{n^3}{3}.$$

We now insert the initial conditions

$$n = 0: \quad u_0 = 1 = C_1,$$
$$n = 1: \quad u_1 = 3 = C_1 + 3C_2 - \frac{1}{2} + \frac{1}{3} \Rightarrow C_2 = \frac{13}{18}.$$

Hence, finally,

$$u_n = 1 + \frac{13n}{18} - \frac{n^2}{2} + \frac{n^3}{3}.$$

If, in (4.8), we have $b_n = b_{1,n} + b_{2,n}$ then our guess for the particular solution should have the form

$$u_{p,n} = u_{p,n}^1 + u_{p,n}^2, \tag{5.2}$$

where, for $i = 1, 2$, $u_{p,n}^i$ would have been our guess if we'd had just $b_n = b_{i,n}$. The next example illustrates this point.

Example 5.4. Let's solve the recurrence

$$u_0 = 1, \quad u_1 = 3, \quad u_{n+2} = 7u_{n+1} - 10u_n + 3^n + 2n \;\; \forall\, n \geq 0.$$

Here $b_{1,n} = 3^n$ and $b_{2,n} = 2n$. The auxiliary equation is $\alpha^2 - 7\alpha + 10 = 0$, which has roots $\alpha_1 = 2, \alpha_2 = 5$. We begin by determining $(u_{p,n})$ and, according to what we've said previously, our guess should be

$$u_{p,n} = u_{p,n}^1 + u_{p,n}^2 = C_3 \cdot 3^n + (C_4 + C_5 n).$$

Insertion into the recurrence gives

$$\begin{aligned} u_{p,n+2} &= 7u_{p,n+1} - 10u_{p,n} + 3^n + 2n \\ \Leftrightarrow C_3 \cdot 3^{n+2} + C_4 + C_5(n+2) &= 7[C_3 \cdot 3^{n+1} + C_4 + C_5(n+1)] - \\ &\quad -10[C_3 \cdot 3^n + C_4 + C_5 n] + 3^n + 2n \\ \Rightarrow 3^n[9C_3 - 21C_3 + 10C_3] + [(C_4 + 2C_5) - 7(C_4 + C_5) + 10C_4] &+ \\ +n[C_5 - 7C_5 + 10C_5] &= 3^n + 2n \\ \Rightarrow -2C_3 = 1 \;\text{ and }\; 4C_4 - 5C_5 = 0 \;\text{ and }\; 4C_5 &= 2 \\ \Rightarrow C_3 = -\frac{1}{2}, \quad C_4 = \frac{5}{8}, \quad C_5 &= \frac{1}{2}. \end{aligned}$$

We have, as usual, $u_{h,n} = C_1 \cdot 2^n + C_2 \cdot 5^n$. Hence, the general solution to the recurrence is

$$u_n = C_1 \cdot 2^n + C_2 \cdot 5^n - \frac{3^n}{2} + \frac{5}{8} + \frac{n}{2}.$$

We now insert the initial conditions

$$n = 0: \quad u_0 = 1 = C_1 + C_2 - \frac{1}{2} + \frac{5}{8} \Rightarrow C_1 + C_2 = \frac{7}{8},$$
$$n = 1: \quad u_1 = 3 = 2C_1 + 5C_2 - \frac{3}{2} + \frac{5}{8}\frac{1}{2} \Rightarrow 2C_1 + 5C_2 = \frac{27}{8}.$$

Gauss elimination yields $C_1 = 1/3$, $C_2 = 13/24$. Hence, finally,

$$u_n = \frac{1}{3} \cdot 2^n + \frac{13}{24} \cdot 5^n - \frac{3^n}{2} + \frac{5}{8} + \frac{n}{2}.$$

The binomial theorem for negative integer exponents. Our goal in forthcoming lectures is to show how the solution of linear recurrences can be carried out in a more systematic manner by means of so-called *generating functions*, and then to illustrate the greater power of such methods by applying them even to other types of problems. A necessary starting point for this discussion is to extend the binomial theorem to situations where the exponent is not a positive integer. We are primarily interested in the case where the exponent is a negative integer, though even a rational exponent will make an appearance when we discuss the *Catalan numbers*. But first, for negative integer exponents, we have the following theorem:

Theorem 5.5. *Let* $n \in \mathbb{N}$ *and* $x \in (-1, 1)$. *Then*

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k. \tag{5.3}$$

We will give three different proofs of this theorem, all of which introduce techniques that are useful to know and that will have applications later on. The first two proofs each start from the observation that, if $|x| < 1$, then

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k. \tag{5.4}$$

Indeed, this should be familiar to you as a formula for the sum of an infinite, convergent geometric series.

First Proof of Theorem 5.5. Raise both sides of (5.4) to the n:th power:

$$(1-x)^{-n} = \left(\frac{1}{1-x}\right)^n = \left(\sum_{k=0}^{\infty} x^k\right)^n = \cdots = \sum_{k=0}^{\infty} c_k x^k,$$

where c_k is the number of ways one can get a term of x^k when one multiplies out the product of n factors. To get x^k one must take x^{k_i} from the i:th factor, $i = 1, 2, \ldots, n$, such that $k_1 + k_2 + \cdots + k_n = k$. Hence, c_k is just the number of solutions to the equation

$$k_1 + k_2 + \cdots + k_n = k, \quad k_i \in \mathbb{N}_0.$$

But from Example 2.11 we know that the number of such solutions is $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$, v.s.v. $\square$

We give the remaining two proofs next time ...