Sixth Lecture: 14/4

Second Proof of Theorem 5.5. Differentiate both sides of (5.4) $n-1$ times. The LHS will become $(n-1)!(1-x)^{-n}$. The RHS can be differentiated termwise[1]. Each of the terms $1, x, \ldots, x^{n-2}$ will collapse to zero after $n-1$ differentiations. For any $k \geq n-1$, $\frac{d^{n-1}}{dx^{n-1}}(x^k) = k(k-1)\ldots(k-n+2)x^{k-(n-1)}$. Thus

$$(n-1)\cdot(1-x)^{-n} = \sum_{k=n-1}^{\infty} k(k-1)\ldots(k-n+2)\, x^{k-(n-1)}.$$

Divide both sides by $(n-1)!$ and change the summation index from k to $l := k-(n-1)$. We get

$$(1-x)^{-n} = \sum_{l=0}^{\infty} \frac{(l+n-1)(l+n-2)\ldots(l+1)}{(n-1)!}\, x^l.$$

The coefficient of x^l is (see (1.9)) just $\binom{l+n-1}{n-1}$, which is the same (see (1.10)) as $\binom{l+n-1}{l}$. $\square$

Third proof of Theorem 5.5. In (5.3) substitute $y := -x$, $t := -n$. Then it becomes

$$\begin{aligned}
(1+y)^t &= \sum_{k=0}^{\infty} \frac{(-t+k-1)(-t+k-2)\ldots(-t+1)(-t)}{k!}(-y)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k t(t-1)\ldots(t-k+2)(t-k+1)}{k!}(-1)^k y^k \\
&= \sum_{k=0}^{\infty} \frac{t(t-1)\ldots(t-k+2)(t-k+1)}{k!}\, y^k.
\end{aligned}$$

Now, for an arbitrary real number t, let us *define* the so-called *generalized binomial coefficient* $\binom{t}{k}$ as

$$\binom{t}{k} \stackrel{\text{def}}{=} \frac{t(t-1)\ldots(t-k+1)}{k!} \tag{6.1}$$

Thus, (5.3) is equivalent to the statement that, if $t \in \mathbb{Z}_{\leq 0}$ then

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k. \tag{6.2}$$

Note that, if instead $t \in \mathbb{Z}_{\geq 0}$ then (6.2) is just the same as (2.1) since, when t is a non-negative integer, a factor in the numerator of (6.1) will be zero once $k > t$ and so the infinite sum will collapse to a finite sum for $k = 0, 1, \ldots, t$.

The point, now, however, is that both sides of (6.2) make sense for any $t \in \mathbb{R}$, provided $|x| < 1$. And, in fact, they are then always equal: this follows by an application of Taylor's theorem to the function $f(x) = (1+x)^t$. Recall that Taylor's theorem

[1] Since the series converges uniformly for all x such that $|x| < 1-\delta$, for any $\delta > 0$.

states that, if $f(x)$ is infinitely differentiable[2] in a neighbourhood of $x = 0$, then in this neighbourhood one has

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k. \tag{6.3}$$

So take $f(x) = (1+x)^t$. One computes directly $f^{(k)}(x) = t(t-1)\dots(t-k+1)(1+x)^{t-k}$. At $x = 0$, one has $(1+0)^{t-k} = 1$ for every k. So substituting into (6.3) yields immediately (6.2). □

Remark 6.1. The statement that, for any $t \in \mathbb{R}$ and $|x| < 1$,

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k, \quad \text{where } \tbinom{t}{k} \text{ is as defined in (6.1)}, \tag{6.4}$$

is referred to as the *Generalized Binomial Theorem.* In this course we will mostly only use the version (5.3), which is more convenient when the exponent is a negative integer. However, in at least one place (when we discuss the Catalan numbers), we will employ version (6.4) with $t = 1/2$.

Generating Functions. We begin with the general definition.

Definition 6.2. Let $(u_n)_{n=1}^{\infty}$ be a sequence of complex numbers. The *generating function* $U(x)$ for the sequence is given by the power series $U(x) = \sum_{n=0}^{\infty} u_n x^n$.

One can, of course, complain that this is not a precise definition, since the domain of the function $U(x)$ has not been specified - indeed, the domain will depend on the particular sequence. This is true, but (i) it will not be an issue when we use generating functions in computations, as we shall see - for instance, in all the examples we do the domain $|x| < 1$, on either the real line or in the complex plane, will do (ii) one can get around this issue quite rigorously by considering $U(x)$ as what is called a *formal power series*, i.e.: an element in a *formal power series ring* $\mathbb{C}[[x]]$. I don't want to go into what this means until we have done examples to build intuition but I will remark on it again later.

Example 6.3. Let's re-solve the recurrence

$$u_0 = 1, \quad u_1 = 3, \quad u_{n+2} = 6u_{n+1} - 8u_n \ \ \forall\, n \geq 0.$$

We already know how to do this using the auxiliary equation method: the equation here is $x^2 - 6x + 8 = 0$, with roots $x_1 = 2$, $x_2 = 4$. Hence the solution is of the form

$$u_n = C_1 \cdot 2^n + C_2 \cdot 4^n.$$

Inserting the initial conditions gives

$$n = 0: \quad u_0 = 1 = C_1 + C_2,$$
$$n = 1: \quad u_1 = 3 = 2C_1 + 4C_2,$$

[2] and satisfies some other technical conditions which I don't want to go into. There are well-known examples of infinitely differentiable functions whose Taylor expansions do not converge to the functions themselves, e.g.: $f(x) = e^{-x^2}$.

and hence $C_1 = C_2 = 1/2$. Thus,

$$u_n = \frac{1}{2}(2^n + 4^n). \tag{6.5}$$

We now illustrate how to get the same result using the generating function method. Though it may seem less efficient, the point is that it should also seem less "ad hoc", and should give greater insight into why the solution has the form it does (no "guessing" is involved).

So let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 1$ we can write, firstly,

$$\begin{aligned} U(x) = 1 + \sum_{n=1}^{\infty} u_n x^n = 1 + x \left(\sum_{n=0}^{\infty} u_{n+1}\, x^n \right) \Rightarrow \\ \Rightarrow \sum_{n=0}^{\infty} u_{n+1}\, x^n = \frac{U(x) - 1}{x}. \end{aligned} \tag{6.6}$$

Then, using also $u_1 = 3$ and the recursion,

$$\begin{aligned} U(x) = 1 + 3x + \sum_{n=2}^{\infty} u_n x^n = (1+3x) + x^2 \left(\sum_{n=0}^{\infty} u_{n+2}\, x^n \right) = \\ = (1+3x) + x^2 \left(6 \sum_{n=0}^{\infty} u_{n+1} x^n - 8 \sum_{n=0}^{\infty} u_n x^n \right) = \\ \overset{(6.6)}{=} (1+3x) + x^2 \left(6 \times \frac{U(x)-1}{x} - 8U(x) \right) \Rightarrow \\ \Rightarrow U(x) = (1+3x) + 6x(U(x) - 1) - 8x^2 U(x) \Rightarrow \end{aligned}$$

$$\Rightarrow (1 - 6x + 8x^2)U(x) = 1 - 3x \Rightarrow U(x) = \frac{1-3x}{1-6x+8x^2} = \frac{1-3x)}{(1-2x)(1-4x)}.$$

The next step is to make a so-called *partial fraction decomposition* of the RHS, namely in this case to find constants A and B such that

$$\frac{1-3x}{(1-2x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-4x}.$$

Multiplying through by $(1-2x)(1-4x)$ we have the requirement that

$$\begin{aligned} 1 - 3x = A(1-4x) + B(1-2x) \\ \Rightarrow 1 - 3x = (A+B) + x(-4A - 2B) \\ \Rightarrow A + B = 1 \quad \text{and} \quad 4A + 2B = 3 \\ \Rightarrow A = B = 1/2. \end{aligned}$$

Thus,

$$U(x) = \frac{1}{2}\left[\frac{1}{1-2x} + \frac{1}{1-4x} \right] = \frac{1}{2}[(1-2x)^{-1} + (1-4x)^{-1}].$$

But then, using (5.3),

$$U(x) = \frac{1}{2}\left[\sum_{k=0}^{\infty} (2x)^k + \sum_{k=0}^{\infty} (4x)^k \right].$$

By comparing the coefficients of x^n, which by definition is equal to u_n on the LHS, we get, as in (6.5),

$$u_n = \frac{1}{2}(2^n + 4^n), \quad \text{v.s.v.}$$