Seventh Lecture: 18/4

Example 7.1. Let's re-do Example 4.6 using generating functions. The recurrence was

$$u_0 = 1, \quad u_1 = 2, \quad u_{n+2} = 6u_{n+1} - 9u_n \ \ \forall\, n \geq 0.$$

The auxiliary equation had a repeated root $x_{1,2} = 3$ and the solution was

$$u_n = \left(1 - \frac{n}{3}\right) 3^n. \tag{7.1}$$

Now instead let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 1$ we can write, firstly,

$$U(x) = 1 + \sum_{n=1}^{\infty} u_n x^n = 1 + x \left(\sum_{n=0}^{\infty} u_{n+1}\, x^n \right) \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} u_{n+1}\, x^n = \frac{U(x) - 1}{x}. \tag{7.2}$$

Then, using also $u_1 = 2$ and the recursion,

$$\begin{aligned}
U(x) = 1 + 2x + \sum_{n=2}^{\infty} u_n x^n &= (1 + 2x) + x^2 \left(\sum_{n=0}^{\infty} u_{n+2}\, x^n \right) = \\
&= (1 + 2x) + x^2 \left(6 \sum_{n=0}^{\infty} u_{n+1} x^n - 9 \sum_{n=0}^{\infty} u_n x^n \right) = \\
&\overset{(7.2)}{=} (1 + 2x) + x^2 \left(6 \times \frac{U(x) - 1}{x} - 9U(x) \right) \Rightarrow \\
&\Rightarrow U(x) = (1 + 2x) + 6x(U(x) - 1) - 9x^2 U(x) \Rightarrow
\end{aligned}$$

$$\Rightarrow (1 - 6x + 9x^2) U(x) = 1 - 4x \Rightarrow U(x) = \frac{1 - 4x}{1 - 6x + 9x^2} = \frac{1 - 4x)}{(1 - 3x)^2}.$$

Since the denominator of the rational function has only one repeated factor, we don't need to make a partial fraction decomposition this time, but can go directly to the Binomial Theorem. The difference from Example 6.3 is that we will now be applying (5.3) with $n = -2$ instead. Precisely,

$$U(x) = (1-4x)(1-3x)^{-2} = (1-4x) \left(\sum_{k=0}^{\infty} \binom{2 + k - 1}{k} (3x)^k \right) = (1-4x) \left(\sum_{k=0}^{\infty} (k+1)\, 3^k\, x^k \right).$$

On the RHS, there are two contributions to the coefficient of x^n, depending on whether we multiply by 1 or by $-4x$ from the first factor. In total, we have, as in (7.1),

$$u_n = 1 \cdot (n+1) 3^n - 4 \cdot n 3^{n-1} = \cdots = \left(1 - \frac{n}{3}\right) 3^n, \quad \text{v.s.v.}$$

Example 7.2. Let's re-do Example 5.4 to illustrate how to handle inhomogeneous equations, in which the most general form of the "right-hand side" (i.e.: of b_n in (4.8)) is the sum of an exponential function and a polynomial. The recurrence was

$$u_0 = 1, \quad u_1 = 3, \quad u_{n+2} = 7u_{n+1} - 10u_n + 3^n + 2n \ \ \forall\, n \geq 0.$$

The solution was

$$u_n = \frac{2^n}{3} + \frac{13 \cdot 5^n}{24} - \frac{3^n}{2} + \frac{5}{8} + \frac{n}{2}. \tag{7.3}$$

Now instead let $U(x) := \sum_{n=0}^{\infty} u_n x^n$. Given that $u_0 = 1$ we can write, firstly,

$$U(x) = 1 + \sum_{n=1}^{\infty} u_n x^n = 1 + x \left(\sum_{n=0}^{\infty} u_{n+1}\, x^n \right) \Rightarrow$$
$$\Rightarrow \sum_{n=0}^{\infty} u_{n+1}\, x^n = \frac{U(x) - 1}{x}. \tag{7.4}$$

Then, using also $u_1 = 3$ and the recursion,

$$U(x) = 1 + 3x + \sum_{n=2}^{\infty} u_n x^n = (1 + 3x) + x^2 \left(\sum_{n=0}^{\infty} u_{n+2}\, x^n \right) =$$
$$= (1 + 3x) + x^2 \left(7 \sum_{n=0}^{\infty} u_{n+1} x^n - 10 \sum_{n=0}^{\infty} u_n x^n + \sum_{n=0}^{\infty} 3^n x^n + 2 \sum_{n=0}^{\infty} n x^n \right). \tag{7.5}$$

The last two sums on the RHS of (7.5) come from the inhomogeneity, so let's first concentrate on how to handle these, i.e.: on how to express them as rational functions. The first of them is just a geometric series:

$$\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1 - 3x}. \tag{7.6}$$

The second sum is handled by differentiating both sides of (5.4), which yields to begin with

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k\, x^{k-1}.$$

Multiply both sides by x, rename the index from k to n and extend the sum to $n = 0$ (note that $n = 0$ contributes zero to the sum). This gives

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}. \tag{7.7}$$

We now substitute (7.2), (7.6) and (7.7) into (7.5) and continue:

$$U(x) = (1+3x) + x^2\left(7 \times \frac{U(x)-1}{x} - 10U(x) + \frac{1}{1-3x} + \frac{2x}{(1-x)^2}\right)$$

$$\Rightarrow U(x) = (1+3x) + 7x(U(x)-1) - 10x^2U(x) + x^2\left[\frac{1}{1-3x} + \frac{2x}{(1-x)^2}\right]$$

$$\Rightarrow (1-7x+10x^2)U(x) = 1-4x + \frac{x^2}{1-3x} + \frac{2x^3}{(1-x)^2} = (1-2x)(1-5x)U(x)$$

$$\Rightarrow U(x) = \frac{1-4x}{(1-2x)(1-5x)} + \frac{x^2}{(1-3x)(1-2x)(1-5x)} + \frac{2x^3}{(1-x)^2(1-2x)(1-5x)}$$

$$\Rightarrow U(x) = \frac{(1-4x)(1-3x)(1-x)^2 + x^2(1-x)^2 + 2x^3(1-3x)}{(1-2x)(1-3x)(1-5x)(1-x)^2} \Rightarrow \dots$$

$$\Rightarrow U(x) = \frac{1-9x+28x^2-31x^3+7x^4}{(1-2x)(1-3x)(1-5x)(1-x)^2}.$$

Since the denominator of the rational function has a repeated factor, the partial fraction decomposition takes the form

$$\frac{1-9x+28x^2-31x^3+7x^4}{(1-2x)(1-3x)(1-5x)(1-x)^2} = \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-5x} + \frac{D}{1-x} + \frac{E}{(1-x)^2}$$

$$\Rightarrow 1-9x+28x^2-31x^3+7x^4 = A(1-3x)(1-5x)(1-x)^2 + B(1-2x)(1-5x)(1-x)^2 + C(1-2x)(1-3x)(1-x)^2 + D(1-2x)(1-3x)(1-5x)(1-x) + E(1-2x)(1-3x)(1-5x).$$

Probably the quickest way to determine the various constants is to insert suitable values of x which cause all but one of the terms on the RHS to become zero. For example, inserting $x = 1/2$ gives

$$1-9\left(\frac{1}{2}\right)+28\left(\frac{1}{2}\right)^2-31\left(\frac{1}{2}\right)^3+7\left(\frac{1}{2}\right)^4 = A\left(1-\frac{3}{2}\right)\left(1-\frac{5}{2}\right)\left(1-\frac{1}{2}\right)^2 \Rightarrow \dots \Rightarrow A = \frac{1}{3}.$$

Similar calculations[1] yield $B = -1/2$, $C = 13/24$, $D = 1/8$, $E = 1/2$. Hence,

$$U(x) = \frac{1/3}{1-2x} - \frac{1/2}{1-3x} + \frac{13/24}{1-5x} + \frac{1/8}{1-x} + \frac{1/2}{(1-x)^2}.$$

Now we are ready to apply the Binomial Theorem:

$$U(x) = \frac{1}{3}\sum_{k=0}^{\infty}(2x)^k - \frac{1}{2}\sum_{k=0}^{\infty}(3x)^k + \frac{13}{24}\sum_{k=0}^{\infty}(5x)^k + \frac{1}{8}\sum_{k=0}^{\infty}x^k + \frac{1}{2}\sum_{k=0}^{\infty}(k+1)x^k.$$

Comparing coefficients of x^n, we get

$$u_n = \frac{1}{3}\cdot 2^n - \frac{1}{2}\cdot 3^n + \frac{13}{24}\cdot 5^n + \frac{1}{8} + \frac{n+1}{2},$$

in agreement with (7.3).

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[1]Details left to reader. Note that one can always choose instead to multiply through by a common denominator, gather coefficients and solve the resulting linear system of five equations in five unknowns, using Gauss elimination - it's up to you !

Catalan numbers. So far we have seen how the introduction of generating functions allows us to approach and to understand the solution of linear recurrence relations in a more methodical manner (sometimes at the cost of longer calculations if done by hand, but easily programmable in a computer). Another advantage of generating functions is that they can sometimes shed light even on more complicated looking recurrences. The Catalan numbers provide a classical illustration of this. We first define them via a sequence of three definitions:

Definition 7.3. A *diagonal path* in the two-dimensional integer lattice $\mathbb{Z}^2$ is a path for which each step is of one of the following four types:

$$\begin{aligned}(x, y) &\to (x+1, y+1) \quad \text{(up and to the right)},\\ (x, y) &\to (x+1, y-1) \quad \text{(down and to the right)},\\ (x, y) &\to (x-1, y+1) \quad \text{(up and to the left)},\\ (x, y) &\to (x-1, y-1) \quad \text{(down and to the left)}.\end{aligned}$$

Definition 7.4. A diagonal path in $\mathbb{Z}^2$ is called a *Dyck path* if it satisfies the following two requirements:

(i) every step is to the right

(ii) the path never goes below the x-axis, though it is allowed to touch the x-axis.

Definition 7.5. Let $n \in \mathbb{N}_0$. The *Catalan number* C_n is defined as the number of Dyck paths from $(0, 0)$ to $(2n, 0)$.

One can compute the first few Catalan numbers by hand[2]: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$ etc - see Figure 7 on the homepage. It is probably not at all obvious at this stage that there is a beautiful general formula for this sequence, namely

Theorem 7.6.

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{7.8}$$

We will give two beautiful proofs of this result next day. The first proof will start by deriving a recurrence for C_n and then manipulate the generating function in a very clever way. This could be termed the “algebraic” proof. The second proof will use a beautiful combinatorial/geometric idea and is more or less a “proof-by-picture”. The second proof will be considerably shorter, but both are works of art and will provide their own type of insight.

Remark 7.7. There are a total of $\binom{2n}{n}$ rightward diagonal paths from $(0, 0)$ to $(2n, 0)$, since any such path contains exactly n upward and n downward steps. Hence, (7.8) says that a fraction $\frac{1}{n+1}$ of these paths have the property that they never go under the x-axis.

[2]If you go to `https://oeis.org` and just write in $1, 1, 2, 5$ the database will already recognise the sequence and return with a wealth of information about the Catalan numbers.