Eighth Lecture: 20/4

First proof of Theorem 7.6. We begin by proving the following recurrence for the Catalan numbers:

$$C_0 = 1, \quad C_n = \sum_{m=1}^{n} C_{m-1}C_{n-m} \;\; \forall\, n \geq 1. \tag{8.1}$$

It is obvious that $C_0 = 1$ - there is only one way to get from $(0, 0)$ to $(0, 0)$, namely to do nothing. Let $n \geq 1$ and consider a Dyck path from $(0, 0)$ to $(2n, 0)$. Let $(2m, 0)$ be the first point at which the path returns to the x-axis after leaving the origin: thus $1 \leq m \leq n$.

The part of the path from $(2m, 0)$ to $(2n, 0)$ can be considered a Dyck path of length $2(n-m)$. Hence there are C_{n-m} possibilities for this part of the path.

Consider the part of the path between $(0, 0)$ and $(2m, 0)$. The first step must be up to $(1, 1)$ and the last step down from $(2m, 1)$. By assumption, the path does not touch the x-axis in between $x = 1$ and $x = 2m-1$. Hence the part of the path between $(1, 1)$ and $(2m, 1)$ can be considered a Dyck path of length $2(m-1)$, shifted upward one unit. So there are C_{m-1} possibilities for this part of the path.

In summary, by MP there are $C_{m-1}C_{n-m}$ possibilities for a Dyck path of length $2n$ which first returns to the x-axis after $2m$ steps. Together with AP, this proves (8.1). Note that the argument is summarized in Figure 8.1 on the homepage.

We now use the generating function method to prove Theorem 7.6. Let $F(x) := \sum_{n=0}^{\infty} C_n x^n$. Observe that $xF(x) = \sum_{n=0}^{\infty} C_n x^{n+1} = \sum_{n=1}^{\infty} C_{n-1}x^n$. Now,

$$[xF(x)][F(x)] = \left[\sum_{n=1}^{\infty} C_{n-1}x^n\right]\left[\sum_{n=0}^{\infty} C_n x^n\right] =$$

$$= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n} C_{m-1}C_{n-m}\right)x^n \overset{(8.1)}{=} \sum_{n=1}^{\infty} C_n x^n$$

$$= \sum_{n=0}^{\infty} C_n x^n - C_0 x^0 = F(x) - 1.$$

Thus

$$x[F(x)]^2 - F(x) + 1 = 0.$$

This can be considered a quadratic equation for $F(x)$ and hence, using the usual formula for the roots of a quadratic equation[1],

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

[1]This formula is derived by completing squares, i.e.: by pure algebraic manipulation of the equation. These manipulations are valid in any *field*, hence in particular in the field of formal power series $\mathbb{C}[[x]]$, in which the function $F(x)$ lies.

Note that if we took the plus sign on the RHS, then it would not converge as $x \to 0$. So the minus sign must be correct and we conclude that

$$F(x) = \frac{1}{2x}\left(1 - (1-4x)^{1/2}\right).$$

We now apply the Generalized Binomial Theorem, eq. (6.4) in the notes, to get

$$F(x) = \frac{1}{2x}\left(1 - \sum_{k=0}^{\infty}\binom{1/2}{k}(-4x)^k\right) = -\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{(-4)^k}{2}x^{k-1}.$$

By definition, C_n is the coefficient of x^n in this expression, hence

$$\begin{aligned} C_n = -\binom{1/2}{n+1}\frac{(-4)^{n+1}}{2} &= -\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2n-1}{2}\right)}{(n+1)!}\,\frac{(-1)^{n+1}4^{n+1}}{2} \\ &= (-1)\times\frac{(-1)^n\times 1\times 3\times\cdots\times(2n-1)}{2^{n+1}\,(n+1)!}\times\frac{(-1)^{n+1}4^{n+1}}{2} \\ &= \frac{1\times 3\times\cdots\times(2n-1)}{(n+1)!}\times 2^n. \end{aligned}$$

Now the trick is to multiply above and below by $2^n\,n!$ and note that $2^n\,n! = 2\times 4\times 6\times\cdots\times 2n$. Thus,

$$\begin{aligned} C_n &= [1\times 3\times\cdots\times(2n-1)][2\times 4\times\cdots\times 2n]\times\frac{2^n}{(2^n\,n!)\times(n+1)!} \\ &= \frac{(2n)!}{n!\,(n+1)!} = \frac{1}{n+1}\,\frac{(2n)!}{n!\,n!} = \frac{1}{n+1}\binom{2n}{n}, \quad \text{v.s.v.} \end{aligned}$$

□

Second proof of Theorem 7.6. First note that

$$\binom{2n}{n-1} = \frac{(2n)!}{(n-1)!\,(n+1)!} = \frac{(2n)!}{n!\,n!}\times\frac{n}{n+1} = \binom{2n}{n}\times\frac{n}{n+1},$$

so in order to prove Theorem 7.6, it suffices to prove that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}. \tag{8.2}$$

Note that $\binom{2n}{n}$ is the total number of rightward diagonal paths from $(0,\,0)$ to $(2n,\,0)$, as already noted in Remark 7.7. Hence we need to show that exactly $\binom{2n}{n-1}$ of these paths go under the x-axis at least once. To accompish this, we will describe one-to-one correspondences between sets of paths as follows (all paths are assumed to be directed rightwards):

$$\begin{aligned} \{\text{Diagonal paths from } (0,\,0) \text{ to } (2n,\,0) \text{ which go under the } x\text{-axis at least once}\} \\ \Updownarrow \\ \{\text{Diagonal paths from } (0,\,1) \text{ to } (2n,\,1) \text{ which meet the } x\text{-axis at least once}\} \\ \Updownarrow \\ \{\text{All diagonal paths from } (0,\,-1) \text{ to } (2n,\,1)\}. \end{aligned}$$

Note that this will suffice: the third set clearly contains $\binom{2n}{n-1}$ paths, since any path in this set contains $n+1$ up-steps and $n-1$ down-steps. The 1-1 correspondence from the first to the second set is also simple to describe, namely just move an entire path one unit upwards. The clever idea is how to get a 1-1 correspondence between the second and third sets.

The idea is called *Andre's Reflection Principle*. Consider a diagonal path from $(0,\,1)$ to $(2n,\,1)$ which meets the x-axis at least once. Let $(k,\,0)$ be the point at which it meets the x-axis for the first time. Now take the part of the path from $(0,\,1)$ to $(k,\,0)$ and *reflect* it in the x-axis. This procedure yields a diagonal path from $(0,\,-1)$ to $(2n,\,1)$, namely the reflection of the original path up to $(k,\,0)$ followed by the original path thereafter. It is then just a matter of realising that *every* diagonal path from $(0,\,-1)$ to $(2n,\,1)$ can be uniquely obtained in this manner, i.e.: by Andre reflection of a unique diagonal path $(0,\,1) \to (2n,\,1)$ which meets the x-axis at least once.

The ideas are summarized in Figure 8.2 on the homepage. □

Multivariable recurrences. We will give some examples of recurrences involving doubly-infinite sequences, that is sequences indexed by two variables $(u(i,\,j))_{i,\,j=0}^{\infty}$. Indeed, one can find examples with an arbitrary number of variables. In general, rigorous mathematical analysis becomes more difficult as the number of variables is increased, with the jump from one to two variables already constituting a quantum leap. This is roughly analogous to the jump from one- to multi-variable calculus, or from ODEs to PDEs. In this course, we just give a few examples which arise naturally in a combinatorial context and in connection to things we have already discussed in earlier lectures. As well as Examples 8.1 and 8.2 below, see Example 9.2 and the discussion on Ramsey numbers in Lecture 10. Further examples can be found amongst the exercises in Biggs, Chapters 12 and 19.

Example 8.1. Let $(c_{n,\,k})_{n,\,k=0}^{\infty}$ be defined by

$$c(0,\,0) = 1, \quad c(0,\,k) = 0 \;\forall\; k > 0, \quad c(n,\,k) = c(n-1,\,k) + c(n-1,\,k-1) \;\forall n \geq 1.$$

Then $c(n,\,k) = \frac{n(n-1)\ldots(n-k+1)}{k!}$. I have no idea how one could arrive at this insight by "algebraic methods". Instead, one just has to realise that the binomial coefficients $\binom{n}{k}$ satisfy exactly the same recurrence, because of Pascal's identity (2.2).

In Lecture 2, we introduced the study of "Balls and Bins" and derived formulae (Propositions 2.8 and 2.9) for the number of ways to place balls into *distinguishable* bins. We now turn to the case where the bins are indistinguishable. It turns out that this case is much harder, and the best we can do is derive fairly simple-looking (but not so simple so as to yield nice explicit formulae) recurrences for the number of ways to distribute the balls in such a way that *no bin is left empty*. There are two cases, depending on whether the balls are distinguishable or not. The first case is Example 8.2 below, the second will be Example 9.2 next time.

Example 8.2. The *Stirling number* $S(n,\,k)$ is defined to be the number of ways to distribute n distinguishable balls among k indistinguishable bins in such a way that no bin is left empty. Clearly, the last requirement implies that $S(n,\, k) = 0$ whenever $k > n$. For other cases, we have the following result, which is Theorem 12.1 in Biggs:

Theorem 8.3.

$$S(n,\, 1) = S(n,\, n) = 1,$$
$$S(n,\, k) = S(n-1,\, k-1) + k \cdot S(n-1,\, k), \quad \textit{whenever}\ 2 \le k \le n-1. \tag{8.3}$$

Proof. $S(n,\, 1) = 1$ since if we have only one bin then it must receive every ball. Similarly, $S(n,\, n) = 1$ since if we have as many bins as balls, and no bin is to be left empty, then each bin must receive exactly one ball. It doesn't matter which bin gets which ball, since the bins are indistinguishable.

Now consider a general pair $(n,\, k)$. Since the balls are distinguishable, we can isolate a particular ball, call it b, and consider two cases:

CASE 1: Ball b is placed in a bin on its own. It doesn't matter which bin, since the latter are identical. It then remains to distribute the remaining $n-1$ balls among $k-1$ bins while not leaving any bin empty. By definition, this can be done in $S(n-1,\, k-1)$ ways.

CASE 2: Ball b is not on its own. In this case it matters where we place ball b - since the balls are distinguishable, it matters which balls it is placed together with. So we begin by distributing the remaining $n-1$ balls into the k bins in such a way that no bin is left empty. This can be done in $S(n-1,\, k)$ ways. There are then k choices for the bin to receive ball b and, since every bin already has at least one ball, it matters where we put ball b. By MP, there are a total of $k \cdot S(n-1,\, k)$ ways to distribute the n balls in Case 2.

Finally, an application of AP yields (8.3). □