Analytic function theory

Basic analysis in one complex variable

Lecture notes

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1 Möbius transformations and the Riemann sphere

There is a deep connection between circles and lines in the plane $\mathbb{C} \simeq \mathbb{R}^2$, and any circle or line can be mapped to any other circle or line by a so called Möbius transformation. By "adding" a point to $\mathbb{C}$ in a certain way one obtains a new space $\widehat{\mathbb{C}}$, the Riemann sphere, which indeed has many properties in common with an ordinary sphere. Möbius transformations are then certain natural bijective maps from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

One may visualize the Riemann sphere $\widehat{\mathbb{C}}$ and its connection to the plane $\mathbb{C}$ geometrically as follows. Consider a sphere in $\mathbb{R}^3$ with two antipodal points; a south pole s and a north pole n. The corresponding equator is a circle in a certain plane which we identify with $\mathbb{C}$ in such a way that the center of the circle is the origin in $\mathbb{C}$. We can then define a bijective map π from the sphere minus the north pole to $\mathbb{C}$: Given a point $p \neq n$ on the sphere there is a well-defined line in $\mathbb{R}^3$ through p and n. This line intersects $\mathbb{C}$ in a unique point q; the map π is then defined by setting $\pi(p) = q$. This map π, which is a version of the Stereographic projection, has many interesting geometric features. For instance, it explains the connection between circles and lines in $\mathbb{C}$. Indeed, circles on the sphere (i.e., the intersection of the sphere and a plane in $\mathbb{R}^3$) not going through the north pole are mapped by π to circles in $\mathbb{C}$ whereas circles going through the north pole are mapped to lines in $\mathbb{C}$.

1.1 Circles and lines

A line ℓ in $\mathbb{C}$ can be described in several ways. For instance, if z_0 and w_0 are two different points on ℓ, then ℓ is the set of complex numbers of the form $z_0 + t(w_0 - z_0)$ where t is a real number; if $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, then this is just the usual parametrization of the line in $\mathbb{R}^2$ through (x_0, y_0) and (u_0, v_0) written in complex notation. Another way to describe ℓ is first to pick a line $\ell^{\perp}$ orthogonal to ℓ and two points α and β on $\ell^{\perp}$ with the same distance to ℓ. Then ℓ is the set of points z such that the distances $|z-\alpha|$ and $|z-\beta|$ are the same, i.e.,

$$\ell = \{z \in \mathbb{C};\ |z-\alpha| = |z-\beta|\} = \{z \in \mathbb{C};\ |z-\alpha|/|z-\beta| = 1\}.$$

Circles in $\mathbb{C}$ may also be described in different ways. Given the center a and the radius r the circle is conveniently described as the set of $z \in \mathbb{C}$ such that $|z-a| = r$, and the circle can be parametrized by the interval $[0, 2\pi)$ using the map

$$t \mapsto a + re^{it} = \operatorname{Re} a + r\cos t + i(\operatorname{Im} a + r \sin t).$$

However, a circle can also be described in a, at least at first sight, less intuitive way. Let $\alpha, \beta \in \mathbb{C}$ and $\lambda \in [0, \infty) \setminus \{1\}$ be given. One can verify (do it, it's a good check that you're fluent in the computation rules in Appendix A) that the equation

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda \tag{1.1}$$

is equivalent to the equation

$$\left|z - \frac{\lambda^2\beta - \alpha}{\lambda^2 - 1}\right| = \frac{\lambda|\alpha - \beta|}{|\lambda^2 - 1|}. \tag{1.2}$$

This means that (1.1) in fact is the equation of a circle with center $(\lambda^2\beta - \alpha)/(\lambda^2 - 1)$ and radius $\lambda|\alpha-\beta|/|\lambda^2-1|$. Conversely, by Exercise 1.2, any circle may be described by an equation like (1.1).

We have thus seen that (1.1) is the equation of a line if $\lambda = 1$ and the equation of a circle if $\lambda \in [0, \infty) \setminus \{1\}$ (the case $\lambda = 0$ is degenerate, the corresponding circle has radius 0, i.e., it is just a point). This observation is a first indication of the correspondence between circles and lines.

1.2 Möbius transformations

We will now look at maps that preserve the set of all circles and lines. For any $a \in \mathbb{C} \setminus \{0\}$ and any $b \in \mathbb{C}$ the maps

$$f_{1,a}(z) = az \quad \text{and} \quad f_{2,b}(z) = z + b$$

map circles to circles and lines to lines since $f_{1,a}$ is a scaling and a rotation about 0 of $\mathbb{C}$ and $f_{2,b}$ is a translation. To be able to change a circle into a line, and vice versa, we introduce $f_3(z) = 1/z$. A priori this is not defined for $z = 0$ but we give it a meaning as follows. Let

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

This requires some explanations; "∞" is just a (suggestive) notation of an abstract point and $\widehat{\mathbb{C}}$ is the set of all complex numbers together with this abstract point. (We don't do any geometric interpretation of $\widehat{\mathbb{C}}$ yet, at this point it is just a set.) We define f_3 to be the map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$f_3(0) = \infty, \quad f_3(\infty) = 0, \quad f_3(z) = 1/z \text{ for } z \in \mathbb{C} \setminus \{0\}.$$

We also extend the maps $f_{1,a}$ and $f_{2,b}$ to be maps of $\widehat{\mathbb{C}}$ by setting $f_{1,a}(\infty) = \infty$ and $f_{2,b}(\infty) = \infty$. In this way we have obtained bijective maps[1], or automorphisms, $f_{1,a}, f_{2,b}, f_3$ of $\widehat{\mathbb{C}}$ and their inverses are of the same type (what are the inverses?). Clearly compositions of such maps are again automorphisms of $\widehat{\mathbb{C}}$ and we obtain a group of automorphisms where the group operation is composition. In view of Exercise 1.3, any composition of maps of the types $f_{1,a}$, $f_{2,b}$, and f_3 is of the form

$$f(z) = \begin{cases} \frac{\alpha z+\beta}{\gamma z+\delta} & , z \neq -\delta/\gamma,\ z \neq \infty \\ \infty & , z = -\delta/\gamma \\ \frac{\alpha}{\gamma} & , z = \infty, \end{cases} \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma \neq 0$. Conversely, see Exercise 1.4, any map of the form (1.3) is a composition of maps of the types $f_{1,a}$, $f_{2,b}$, and f_3. The automorphisms of $\widehat{\mathbb{C}}$ of the form (1.3) are called Möbius transformations.

Theorem 1.1. *Möbius transformations preserve the set of circles and lines in* $\mathbb{C}$*.*

Bevis. Since any Möbius transformation is a composition of maps of the types $f_{1,a}$, $f_{2,b}$, and f_3, and maps of the types $f_{1,a}$ and $f_{2,b}$ preserve the set of circles and lines, it is sufficient to check that the map f_3 preserves the set of circles and lines.

Let $\mathcal{C}$ be a circle or line. We know that it can be described by an equation $|z-\alpha|/|z-\beta| = \lambda$ for some $\alpha, \beta \in \mathbb{C}$ and $\lambda > 0$. Then

$$\begin{aligned} f_3(\mathcal{C}) &= \{w = 1/z;\ |z-\alpha|/|z-\beta| = \lambda\} \\ &= \left\{w;\ \frac{|1/w-\alpha|}{|1/w-\beta|} = \lambda\right\}. \end{aligned}$$

But

$$\frac{|1/w-\alpha|}{|1/w-\beta|} = \frac{|1-\alpha w|}{|1-\beta w|} = \frac{|\alpha|}{|\beta|}\frac{|w-1/\alpha|}{|w-1/\beta|}$$

so $f_3(\mathcal{C})$ is again described by the equation $|w-1/\alpha|/|w-1/\beta| = \lambda|\beta|/|\alpha|$, i.e., it is a circle or a line. The cases $\alpha = 0$ or $\beta = 0$ may be checked separately in a similar way. $\square$

[1] These are in fact continuous in the sense described in Appendix B.

1.3 Visualizing Möbius transformations of the Riemann sphere

Let Σ be the sphere $\{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2=1\}$ and let P be the xy-plane, i.e., $P = \{(x,y,z) \in \mathbb{R}^3;\ z=0\}$. We denote the north pole $(0,0,1)$ by n and the south pole $(0,0,-1)$ by s; we will define two stereographic projections, $\pi_n \colon \Sigma \setminus \{n\} \to P$ and $\pi_s \colon \Sigma \setminus \{s\} \to P$.

Given $(x,y,z) \in \Sigma \setminus \{n\}$ consider the line in $\mathbb{R}^3$ through n and (x,y,z); it can be parametrized by $\mathbb{R}$ using the map $t \mapsto (tx, ty, 1+t(z-1))$. This line intersects the plane P in a point that we denote by $\pi_n(x,y,z)$. We can obtain an explicit expression for $\pi_n(x,y,z)$ as follows. The line intersects P when t is such that $1+t(z-1)=0$, that is when $t = 1/(1-z)$. Putting this value of t into the parametrization of the line we see that

$$\pi_n(x,y,z) = \big(x/(1-z), y/(1-z), 0\big).$$

Similarly, given $(x,y,z) \in \Sigma \setminus \{s\}$ we define $\pi_s(x,y,z)$ to be the point of intersection between P and the line through s and (x,y,z), and we get

$$\pi_s(x,y,z) = \big(x/(1+z), y/(1+z), 0\big).$$

One can check (Exercise 1.5)that the inverses of π_n and π_s are given by

$$\pi_n^{-1}(x,y,0) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right) \quad \text{and} \tag{1.4}$$

$$\pi_s^{-1}(x,y,0) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-x^2-y^2+1}{1+x^2+y^2}\right), \tag{1.5}$$

respectively.

We can identify the plane P and $\mathbb{C}$ in two different ways using the maps $\mathrm{id}\colon P \to \mathbb{C}$ and $\overline{\mathrm{id}}\colon P \to \mathbb{C}$ defined by $\mathrm{id}(x,y,0) = x+iy$ and $\overline{\mathrm{id}}(x,y,0) = x-iy$, respectively. For $w = x+iy \neq 0$ we now have

$$\begin{aligned}
\overline{\mathrm{id}} \circ \pi_s \circ \pi_n^{-1} \circ \mathrm{id}^{-1}(w) &= \overline{\mathrm{id}} \circ \pi_s \circ \pi_n^{-1}(x,y,0) = \overline{\mathrm{id}} \circ \pi_s \left(\frac{2x}{1+|w|^2}, \frac{2y}{1+|w|^2}, \frac{|w|^2-1}{1+|w|^2}\right) \\
&= \overline{\mathrm{id}}\left(\frac{x}{|w|^2}, \frac{y}{|w|^2}, 0\right) = \frac{x}{|w|^2} - i\frac{y}{|w|^2} = \frac{\bar{w}}{|w|^2} \\
&= \frac{1}{w}.
\end{aligned}$$

This means that if we identify $\widehat{\mathbb{C}}$ with Σ by identifying $w = x+iy$ with $\pi_n^{-1}(x,y,0)$ and ∞ with n, then the map f_3 defined above is the automorphism of Σ defined by $(x,y,z) \mapsto (x,-y,-z)$, i.e., reflection in the x-axis.

Via this identification of $\widehat{\mathbb{C}}$ with Σ we get a geometric interpretation of $\widehat{\mathbb{C}}$ as a sphere which is called the Riemann sphere. Moreover, since we already have geometric interpretations of maps of the types $f_{1,a}$ and $f_{2,b}$, defined above, and any Möbius transformation is a composition of such maps and f_3-maps, we have a way of visualizing any Möbius transformation as a certain automorphism of the Riemann sphere.

Exercises

1.1 Show that Equation (1.1) and Equation (1.2) are equivalent.

1.2 Let $a \in \mathbb{C}$ and $r \geq 0$. Show that there are $\alpha, \beta \in \mathbb{C}$ and $\lambda \in [0,\infty) \setminus \{1\}$ such that

$$a = \frac{\lambda^2\beta - \alpha}{\lambda^2 - 1} \quad \text{and} \quad r = \frac{\lambda|\alpha - \beta|}{|\lambda^2 - 1|}.$$

1.3 Show, for instance using induction, that any composition of maps of the types $f_{1,a}$, $f_{2,b}$, and f_3 is of the form (1.3).

1.4 Show that a Möbius transformation is a composition of maps of the type $f_{1,a}$, $f_{2,b}$, and f_3.

1.5 Check that the inverses of π_n and π_s are given by (1.4) and (1.5), respectively.

1.6 Compute $f_3(\ell)$ where ℓ is the line $\{1 + it;\, t \in \mathbb{R}\}$.

1.7 Find a Möbius transformation that maps the real line to the unit circle.

1.8 Let $f(z) = (z-1)/(z+1)$. Find the fixed points of f, i.e., find all points z on $\widehat{\mathbb{C}}$ such that $f(z) = z$. Compute the images of the unit circle and the real axis respectively.

2 Two-variable real calculus in complex notation

We will recall some results and notions from calculus in two real variables and rewrite them in a notation that is suitable for complex analysis.

2.1 Differentiability

Let f be a (possibly complex-valued) function defined in an open subset $\Omega \subset \mathbb{C} \simeq \mathbb{R}^2$. Recall from calculus that f is differentiable at a point $z_0 = x_0 + iy_0 \in \Omega$ if there are (possibly complex) numbers A and B such that

$$f(x,y) = f(x_0,y_0) + A(x - x_0) + B(y - y_0) + o(\sqrt{(x-x_0)^2 + (y-y_0)^2}). \tag{2.1}$$

If there are such A and B then the partial derivatives of f at the point (x_0, y_0) exist and $A = \frac{\partial f}{\partial x}(x_0,y_0)$, $B = \frac{\partial f}{\partial y}(x_0,y_0)$. By setting $C := (A - iB)/2$ and $D := (A+iB)/2$ we can rewrite (do it!) (2.1) as

$$f(z) = f(z_0) + C(z - z_0) + D(\bar{z} - \bar{z}_0) + o(|z - z_0|).$$

Thus, if we define partial derivatives with respect to z and $\bar{z}$ by setting

$$\frac{\partial f}{\partial z}(z_0) := \frac{1}{2}\Big(\frac{\partial f}{\partial x}(x_0,y_0) - i\frac{\partial f}{\partial y}(x_0,y_0)\Big), \tag{2.2}$$

$$\frac{\partial f}{\partial \bar{z}}(z_0) := \frac{1}{2}\Big(\frac{\partial f}{\partial x}(x_0,y_0) + i\frac{\partial f}{\partial y}(x_0,y_0)\Big), \tag{2.3}$$

then (2.1) may be written as

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)(\bar{z} - \bar{z}_0) + o(|z - z_0|). \tag{2.4}$$

Proposition 2.1. *Let $f\colon \Omega \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$. Then*

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \tag{2.5}$$

exists if and only if $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. Moreover, if (2.5) exists then it is equal to $\frac{\partial f}{\partial z}(z_0)$.

Proof. Assume first that $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. Setting $z = z_0 + h$ in (2.4) we get

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{\partial f}{\partial z}(z_0) + \frac{o(|h|)}{h},$$

and since $o(|h|)/h \to 0$ as $h \to 0$ we see that (2.5) exists and equals $\frac{\partial f}{\partial z}(z_0)$.

Conversely, assume that (2.5) exists. Setting $z = z_0 + h$ in (2.4) and rewriting we get

$$\frac{\partial f}{\partial \bar{z}}(z_0)\frac{\bar{h}}{h} = \frac{f(z_0+h) - f(z_0)}{h} - \frac{\partial f}{\partial z}(z_0) + \frac{o(|h|)}{h}.$$

By assumption the limit as $h \to 0$ of the right-hand side exists and so the limit of the left-hand side exists; recall (see, e.g., Appendix B) that this means that for every sequence $\{h_n\}$

of complex numbers converging to 0, the sequence $\frac{\partial f}{\partial \bar{z}}(z_0)\frac{\bar{h}_n}{h_n}$ converges to one fixed number. If we let $h \to 0$ along the real axis, e.g., we choose $h_n = 1/n$, then

$$\frac{\partial f}{\partial \bar{z}}(z_0)\frac{\bar{h}_n}{h_n} \to \frac{\partial f}{\partial \bar{z}}(z_0), \quad n \to \infty.$$

On the other hand, if let $h \to 0$ along the imaginary axis, e.g., choosing $h_n = i/n$, then

$$\frac{\partial f}{\partial \bar{z}}(z_0)\frac{\bar{h}_n}{h_n} \to -\frac{\partial f}{\partial \bar{z}}(z_0), \quad n \to \infty.$$

Hence, $\frac{\partial f}{\partial \bar{z}}(z_0) = -\frac{\partial f}{\partial \bar{z}}(z_0)$ and so $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. $\square$

A function $f\colon \Omega \to \mathbb{C}$ such that the limit (2.5) exists is said to be *complex differentiable* at z_0. We notice that if f is complex differentiable at z_0 then f is continuous at z_0, Exercise 2.4. We notice also that the proposition above in this terminology says that a differentiable function is complex differentiable if and only $\partial f/\partial \bar{z} = 0$. We will see in Theorem 3.5 below that a function f that is complex differentiable, but a priori not necessarily differentiable, in fact still satisfies $\partial f/\partial \bar{z} = 0$.

2.2 Curves and contour integrals

Let $\Omega \subset \mathbb{C} \simeq \mathbb{R}^2$ be open. By a *parametrized curve* in Ω we mean a continuous map $\gamma\colon I \to \Omega$, where $I \subset \mathbb{R}$ is an interval. A parametrized curve γ is said to be *simple* if γ is injective and *closed* if the image $\gamma(I)$ is a closed subset of $\mathbb{C}$ and $\lim_{t\to a^+}\gamma(t) = \lim_{t\to b^-}\gamma(t)$, where a and b are the left and right end points (possibly $-\infty$ and ∞) of the interval I, respectively. A subset $\mathcal{C}$ of Ω is called a *curve* in Ω if it is the image of a parametrized curve in Ω, i.e., if $\mathcal{C} = \gamma(I)$ for some parametrized curve γ in Ω. A curve is simple and closed, respectively, if it correspond to a parametrized curve which is simple and closed, respectively. We will refer to a parametrized curve corresponding to a curve $\mathcal{C}$ as a parametrization of $\mathcal{C}$.

General curves may look very non-curve-like, for instance the so called Peano curve (google it) fills the 2-dimensional square $[0,1] \times [0,1]$. In these notes such curves will not be of interest and we always assume some extra conditions to ensure that our curves are curve-like. Specifically, we often assume that our parametrized curves are C^1-smooth, or simply smooth, meaning that γ is continuously differentiable, or at least that they are piecewise C^1-smooth. This means that there are points $\cdots t_{k-1} < t_k < t_{k+1} < \cdots$ in I such that the restriction, $\gamma\restriction_{(t_k,t_{k+1})}$, of γ to the sub-interval (t_k, t_{k+1}) of I is C^1-smooth for all k. We say that a subset $\mathcal{C}$ of Ω is a C^1-smooth (piecewise C^1-smooth) curve if $\mathcal{C} = \gamma(I)$ for some C^1-smooth (piecewise C^1-smooth) parametrized curve γ.

Example 2.2. Let $\gamma(t) = e^{it} = \cos t + i \sin t$, $0 \le t \le 2\pi$. Then γ is a smooth closed curve in $\mathbb{C}$. The image of γ is the unit circle $\{z \in \mathbb{C};\ |z| = 1\}$, which is smooth, closed, and simple. We can make γ simple too by restricting t to $[0, 2\pi)$.

Example 2.3. The set $\{z = x + iy \in \mathbb{C};\ y = 0, -1 \le x \le 1\} \cup \{z \in \mathbb{C};\ |z| = 1, \operatorname{Im} z > 0\}$, which is the upper part of the unit circle together with the interval $[-1, 1]$, is a simple closed piecewise smooth curve.

A simple closed curve in $\mathbb{C}$ divides $\mathbb{C}$ into two connected pieces, one bounded piece, called the interior of the curve, and one unbounded. This follows in general from the Jordan curve theorem which is a rather deep result. However, for most curves appearing in these notes this will be obvious. An *orientation* of a curve $\mathcal{C}$ is a choice of traveling direction along $\mathcal{C}$. A simple closed curve $\mathcal{C}$ is said to be *positively oriented* if the traveling direction is such that one has the interior of $\mathcal{C}$ on the left-hand side; it is said to be *negatively oriented* if the interior is on the right-hand side. When we say that $\gamma\colon I \to \mathbb{C}$ is a parametrization of an oriented curve $\mathcal{C}$ we always mean that $\gamma(t)$ runs through $\mathcal{C}$ in the direction given by the orientation as t runs through I from left to right.

Let $\mathcal{C} \subset \mathbb{C} \simeq \mathbb{R}^2$ be a C^1-smooth simple oriented curve and let P and Q be continuous functions defined on $\mathcal{C}$. We recall from calculus that the line integral of the differential form[2] $Pdx + Qdy$ is defined as follows. Let $\gamma\colon I \to \mathbb{R}^2$ be a simple C^1-smooth parametrization of $\mathcal{C}$ (tacitly assumed compatible with the orientation). We write $\gamma(t) = (x(t), y(t))$. Then

$$\int_{\mathcal{C}} Pdx + Qdy := \int_{t\in I} \big(P\big(\gamma(t)\big)x'(t) + Q\big(\gamma(t)\big)y'(t)\big)\,dt. \tag{2.6}$$

This is well defined since we know from calculus that the right-hand side is independent of the choice of simple parametrization γ of $\mathcal{C}$. More generally, if $\mathcal{C}$ is a piecewise smooth curve, which is also piecewise oriented, then we divide $\mathcal{C}$ into pairwise disjoint curve-pieces $\mathcal{C}_j$ that are smooth, simple, and oriented, and we set

$$\int_{\mathcal{C}} Pdx + Qdy := \sum_j \int_{\mathcal{C}_j} Pdx + Qdy.$$

Let us rewrite line integrals in $\mathbb{R}^2 \simeq \mathbb{C}$ in complex notation. If $z = x + iy$ then $\bar{z} = x - iy$ and it is natural to define

$$dz := dx + idy \quad \text{and} \quad d\bar{z} := dx - idy. \tag{2.7}$$

It follows that $dx = (dz + d\bar{z})/2$ and $dy = (dz - d\bar{z})/2i$ and a simple computation shows that we can write a differential form $Pdx + Qdy$ as

$$Pdx + Qdy = \frac{1}{2}(P - iQ)dz + \frac{1}{2}(P + iQ)d\bar{z}.$$

In particular, if f is a differentiable function we have by (2.2) and (2.3) that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}, \tag{2.8}$$

cf. Exercise 2.7 below. If now $\gamma(t) = x(t) + iy(t)$ is a C^1-smooth function $I \to \mathbb{C} \simeq \mathbb{R}^2$, then $\overline{\gamma}(t) := x(t) - iy(t)$ is a C^1-smooth function and we can rewrite the right-hand side of (2.6) as

$$\int_{t\in I} \left(\frac{P\big(\gamma(t)\big) - iQ\big(\gamma(t)\big)}{2}\gamma'(t) + \frac{P\big(\gamma(t)\big) + iQ\big(\gamma(t)\big)}{2}\overline{\gamma}'(t) \right) dt,$$

which we define to be the integral of the differential form $(P - iQ)dz/2 + (P + iQ)d\bar{z}/2$ along the curve $\gamma(I)$.

To sum up, any differential form $Pdx+Qdy$ may be written $fdz+gd\bar{z}$, where $f = (P-iQ)/2$ and $g = (P+iQ)/2$, and, conversely, any differential form $fdz+gd\bar{z}$ may be written $Pdx+Qdy$ by setting $P = f + g$ and $Q = i(f - g)$. Moreover, the *contour integral* of $fdz + gd\bar{z}$ along an oriented curve $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} fdz + gd\bar{z} := \int_{\mathcal{C}} (f + g)dx + i(f - g)dy,$$

where the right-hand side is the ordinary line integral as defined in calculus.

[2]We presume some basic operative knowledge about differential forms.

Example 2.4. Let $\mathcal{C} = \{z \in \mathbb{C}; |z| = r\}$ with positive orientation (i.e., run through counter-clockwise) and let $f(z) = 1/z$; notice that f is defined for $z \in \mathcal{C}$. Compute the contour integral $\int_{\mathcal{C}} f dz$.

Solution: Choose $\gamma(t) = re^{it} = r\cos t + i\sin t$, $0 \le t < 2\pi$, as simple parametrization of $\mathcal{C}$. We then have

$$\begin{aligned}\int_{\mathcal{C}} f dz &= \int_{\mathcal{C}} f dx + i f dy = \int_0^{2\pi} f\big(\gamma(t)\big)\big(x'(t) + iy'(t)\big)dt = \int_0^{2\pi} f\big(\gamma(t)\big)\gamma'(t)dt \\ &= \int_0^{2\pi} \frac{-r\sin t + ir\cos t}{r\cos t + ir\sin t}dt = \int_0^{2\pi} idt = 2\pi i.\end{aligned}$$

This computation can be made shorter and more suggestive by noting that if $z = re^{it}$, then $dz = ire^{it}dt$ (show this!), and so

$$\int_{\mathcal{C}} f dz = \int_0^{2\pi} \frac{ire^{it}dt}{re^{it}} = \int_0^{2\pi} idt = 2\pi i.$$

We now recall Green's formula from calculus; it relates the line integral of a differential form along the boundary of an open set Ω to an integral over Ω. More precisely, we have

Theorem 2.5 (Green's formula)**.** *Let Ω be a bounded open subset of $\mathbb{R}^2 \simeq \mathbb{C}$ and assume that its boundary, $\partial\Omega$, is a piecewise C^1-smooth curve, positively oriented. Let P and Q be continuously differentiable functions defined in some open set $U \supset \overline{\Omega}$. Then*

$$\int_{\partial\Omega} P dx + Q dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

In particular, if f is a continuously differentiable function in $U \supset \overline{\Omega}$ it follows that

$$\int_{\partial\Omega} f dz = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar{z}} dxdy. \tag{2.9}$$

In fact, by Green's formula and (2.3) we have

$$\begin{aligned}\int_{\partial\Omega} f dz &= \int_{\partial\Omega} f dx + i f dy = \iint_{\Omega} \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy \\ &= 2i \iint_{\Omega} \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dxdy = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar{z}} dxdy.\end{aligned}$$

Remark 2.6. If $\partial\Omega$ is a union of piecewise smooth positively oriented curves $\partial\Omega_j$, then the conclusion still holds if we construe the left-hand side as the sum of integrals over the boundary parts $\partial\Omega_j$.

Let $\gamma\colon I \to \mathbb{C}$ be a smooth simple parametrized curve and assume that I is bounded. Then the number $\int_I |\gamma'(t)|dt$ is independent of such parametrization and thus only depends on the curve $\mathcal{C} := \gamma(I)$. We call this number the length of $\mathcal{C}$ and denote it by $\ell(\mathcal{C})$. We define the length of a piecewise smooth curve by dividing it into smooth simple pieces and sum up the lengths of these.

Proposition 2.7. *Let $\mathcal{C} \subset \mathbb{C}$ be a piecewise smooth oriented curve and let f be a continuous function defined on $\mathcal{C}$. Then*

$$\left|\int_{\mathcal{C}} f(z)dz\right| \le \sup_{z\in\mathcal{C}} |f(z)| \cdot \ell(\mathcal{C}).$$

Proof. Possibly after dividing $\mathcal{C}$ into suitable pieces we may assume that $\mathcal{C}$ is given by a smooth simple parametrized curve $\gamma\colon I \to \mathbb{C}$. Then we have

$$\begin{aligned}\left|\int_{\mathcal{C}} f(z)dz\right| &= \left|\int_I f\big(\gamma(t)\big)\gamma'(t)dt\right| \le \int_I \left|f\big(\gamma(t)\big)\gamma'(t)\right|dt \le \sup_{t\in I}\left|f\big(\gamma(t)\big)\right| \int_I |\gamma'(t)|dt \\ &= \sup_{z\in\mathcal{C}} |f(z)| \cdot \ell(\mathcal{C}).\end{aligned}$$

The second inequality is well-known from calculus as is the first, at least if $f\big(\gamma(t)\big)\gamma'(t)$ is real for all t. A proof that it holds in general is outlined in Exercise 2.5. □

Exercises

2.1 Let f and g be differentiable functions. Show that

(a) $\dfrac{\partial}{\partial z}\big(af + bg\big) = a\dfrac{\partial f}{\partial z} + b\dfrac{\partial g}{\partial z}$ and $\dfrac{\partial}{\partial \bar z}\big(af + bg\big) = a\dfrac{\partial f}{\partial \bar z} + b\dfrac{\partial g}{\partial \bar z}$ for all complex constants a and b,

(b) $\dfrac{\partial}{\partial z}(f \cdot g) = f\dfrac{\partial g}{\partial z} + g\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial}{\partial \bar z}(f \cdot g) = f\dfrac{\partial g}{\partial \bar z} + g\dfrac{\partial f}{\partial \bar z}$,

(c) $\dfrac{\partial}{\partial z}\Big(\dfrac{f}{g}\Big) = \dfrac{g\partial f/\partial z - f\partial g/\partial z}{g^2}$ and $\dfrac{\partial}{\partial \bar z}\Big(\dfrac{f}{g}\Big) = \dfrac{g\partial f/\partial \bar z - f\partial g/\partial \bar z}{g^2}$ if $g \neq 0$.

2.2 Compute $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \bar z}$ when

$$\text{(a)}\ \ f(z) = z, \qquad \text{(b)}\ \ f(z) = \bar z, \qquad \text{(c)}\ \ f(z) = az^k\bar z^\ell.$$

2.3 Show that $\overline{\left(\dfrac{\partial f}{\partial \bar z}\right)} = \dfrac{\partial \bar f}{\partial z}$ and $\overline{\left(\dfrac{\partial f}{\partial z}\right)} = \dfrac{\partial \bar f}{\partial \bar z}$.

2.4 Show that if f is complex differentiable at z_0, then f is continuous at z_0.

2.5 Let $I \subset \mathbb{R}$ and let $g\colon I \to \mathbb{C}$ be a function. The purpose of this exercise is to outline a proof of the inequality

$$\left|\int_I g(t)dt\right| \le \int_I |g(t)|dt \tag{2.10}$$

given that we know the corresponding inequality for real-valued functions.

(a) Assume that $\int_I g(t)dt$ is a real number, i.e., that $\int_I \operatorname{Im} g(t)dt = 0$, and show (2.10) in this case using computation rule (12) in Appendix A.

(b) Show that there is a $\theta \in \mathbb{R}$ such that $e^{i\theta}\int_I g(t)dt$ is a real number and conclude (2.10) using part (a).

2.6 Compute the integral $\displaystyle\int_{\mathcal{C}} f(z)dz$, where $f(z) = \operatorname{Re} z$ and $\mathcal{C}$ is the oriented curve given by the parametrization $\gamma(t) = t + it^2$, $0 \le t \le 1$. Notice that f is real-valued; is the integral a real number?

2.7 Let $\gamma\colon I \to \mathbb{C}$ be a smooth parametrized curve and let f be a differentiable function in an open set containing $\gamma(I)$. Set $\psi(t) = f \circ \gamma(t)$ and use the Chain rule of calculus to show that $\psi'(t) = \dfrac{\partial f}{\partial z}\big(\gamma(t)\big) \cdot \gamma'(t) + \dfrac{\partial f}{\partial \bar z}\big(\gamma(t)\big) \cdot \overline{\gamma'(t)}$.

2.8 Let $\mathcal{C}$ be a piecewise smooth oriented curve of finite length in $\mathbb{C}$ and let f_n, $n = 1, 2, \ldots$, be continuous functions on $\mathcal{C}$ such that $f_n(z)$ converges uniformly on $\mathcal{C}$ to some function $f(z)$. Show, e.g., using Proposition 2.7, that

$$\lim_{n\to\infty} \int_{\mathcal{C}} f_n(z)dz = \int_{\mathcal{C}} f(z)dz.$$

2.9 Show that $\displaystyle\int_{\partial D(a,r)} (z-a)^k dz = \begin{cases} 2\pi i \text{ if } k = -1, \\ 0 \text{ if } k \neq -1 \end{cases}$; cf. Example 2.4.

3 Holomorphic functions and Cauchy's theorem and formula

We define a holomorphic function as a C^1-smooth function that is complex differentiable and we show some basic properties. We also show that complex differentiability alone implies that the Cauchy-Riemann equation(s) are satisfied. We then prove Cauchy's theorem using Green's formula and show how Cauchy's formula follows from his theorem.

3.1 Definition and basic properties of holomorphic functions

Recall that a function f defined in an open subset Ω of $\mathbb{R}^2 \simeq \mathbb{C}$ is C^1-smooth in Ω, $f \in C^1(\Omega)$, if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at each point $z \in \Omega$ and depend continuously on z. From calculus we know that if f is C^1-smooth in Ω then f is in particular differentiable at each point of Ω.

Definition 3.1. Let Ω be an open subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be a function. Then f is *holomorphic* in Ω if $f \in C^1(\Omega)$ and f is complex differentiable at each point of Ω, i.e., if the limit (2.5) exists for each $z_0 \in \Omega$.

Notice that, by Proposition 2.1, a C^1-smooth function f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$. Moreover, if f is holomorphic then the limit (2.5) is equal to $\partial f/\partial z$ at the point z_0; we will then usually write f' instead $\partial f/\partial z$ and refer to f' as the derivative of f.

Remark 3.2. The requirement $f \in C^1(\Omega)$ is actually superfluous and indeed follows if f is assumed complex differentiable at each point. This is an interesting fact that we will prove below (Goursat's theorem) but, in my opinion, not so important for complex analysis since checking C^1-smoothness often is easier than checking that (2.5) exists everywhere.

Let f and g be holomorphic functions in an open set $\Omega \subset \mathbb{C}$. Then $af+bg$ is holomorphic in Ω for all constants a and b, fg is holomorphic in Ω, and f/g is holomorphic in Ω if g is non-zero. Indeed, it is clear that $af+bg$, fg, and f/g are C^1-smooth and since $\partial f/\partial \bar{z} = \partial g/\partial \bar{z} = 0$ it follows that $\partial(af+bg)/\partial \bar{z} = 0$, that $\partial(fg)/\partial \bar{z} = 0$, and that $\partial(f/g)/\partial \bar{z} = 0$, see Exercise 2.1.

Example 3.3. The function $f(z) = z$ is holomorphic in $\mathbb{C}$ since it is obviously C^1, and complex differentiability is straightforward to check. (It is also straightforward to check that $\partial f/\partial \bar{z} = 0$, cf. Exercise 2.2.) Thus, any polynomial $\sum_{k=0}^n a_k z^k$ is holomorphic in $\mathbb{C}$. Moreover, if $p(z)$ and $q(z)$ are polynomials, then the *rational function* $p(z)/q(z)$ is holomorphic in $\mathbb{C} \setminus \{z;\, q(z) = 0\}$.

If f is holomorphic in Ω and g is holomorphic in an open set containing the image $f(\Omega)$, then the composition $g \circ f$ is holomorphic in Ω. In fact, it is known from calculus that $g \circ f \in C^1(\Omega)$ and as in one-variable real calculus one can show that $g \circ f$ is complex differentiable at each point of Ω. A different argument is suggested in Exercise 3.1 below which also shows that we have the following version of the Chain rule for holomorphic functions:

$$(g \circ f)'(z) = g'\big(f(z)\big) \cdot f'(z).$$

The following result can be seen as a version of the Fundamental theorem of calculus for holomorphic functions.

Proposition 3.4. *Let f be holomorphic in an open set Ω and let $\mathcal{C}$ be an oriented piecewise smooth curve in Ω starting at a and ending at b. Then* $\displaystyle\int_{\mathcal{C}} f'(z)dz = f(b) - f(a)$.

Notice in particular that the contour integral $\displaystyle\int_{\mathcal{C}} f'(z)dz$ only depends on the start and end points of $\mathcal{C}$, not on the particular curve connecting these points.

Proof of Proposition 3.4. Possibly after dividing $\mathcal{C}$ into pieces we may assume that we have a C^1-smooth parametrization $\gamma\colon I \to \mathbb{C}$ of $\mathcal{C}$. Write $\gamma(t) = x(t) + iy(t)$ and let t_0 and t_1 be the left and right endpoints, respectively, of I. Now, since f is holomorphic we have, in view of (2.8), that $f'(z)dz = \partial f/\partial x\, dx + \partial f/\partial y\, dy$ and so

$$\begin{aligned}\int_{\mathcal{C}} f'(z)dz &= \int_{\mathcal{C}} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \int_{t_0}^{t_1}\left(\frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t)\right)dt = \int_{t_0}^{t_1}\frac{d}{dt}f\big(\gamma(t)\big)dt \\ &= f\big(\gamma(t_1)\big) - f\big(\gamma(t_0)\big) = f(b) - f(a).\end{aligned}$$

The third equality follows from the chain rule of calculus (cf. Exercise 2.7) and the fourth equality is the Fundamental theorem of calculus applied to the function $g(t) = f\big(\gamma(t)\big)$. □

One indication that the existence of the limit (2.5) alone has strong consequences is the next result; it shows that a function f that is complex differentiable at a point, but a priori not necessarily differentiable, satisfies $\partial f/\partial \bar{z} = 0$, cf. Proposition 2.1 above and the comment following its proof.

Theorem 3.5 (The Cauchy-Riemann equations)**.** *Let f be a function that is complex differentiable at a point z_0 and write $f(z) = u(z) + iv(z)$. Then the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist at z_0 and*

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0). \tag{3.1}$$

The equations (3.1) are known as the Cauchy-Riemann equations (at z_0) and they are equivalent to the equation $\partial f/\partial \bar{z} = 0$. To see this we recall (2.3) and compute:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\Big(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\Big)\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)\right).\end{aligned}$$

Identifying real and imaginary parts we thus see that $\partial f/\partial \bar{z} = 0$ at z_0 if and only if (3.1) hold.

Proof of Theorem 3.5. We know that the limit (3.1) exists; denote it by $f'(z_0)$. By Lemma 14.10 (b) applied to the function $g(h) := (f(z_0+h) - f(z_0))/h$, the limits $\lim_{h\to 0}\operatorname{Re}\big((f(z_0+h) - f(z_0))/h\big)$ and $\lim_{h\to 0}\operatorname{Im}\big((f(z_0+h) - f(z_0))/h\big)$ exist and

$$f'(z_0) = \lim_{h\to 0}\operatorname{Re}\frac{f(z_0+h) - f(z_0)}{h} + i\lim_{h\to 0}\operatorname{Im}\frac{f(z_0+h) - f(z_0)}{h}. \tag{3.2}$$

If h is real, then $\operatorname{Re}\big((f(z_0+h) - f(z_0))/h\big) = (u(z_0+h) - u(z_0))/h$ and $\operatorname{Im}\big((f(z_0+h) - f(z_0))/h\big) = (v(z_0+h) - v(z_0))/h$, and so if we let $h \to 0$ along the real axis, then the right-hand side of (3.2) equals $\partial u/\partial x + i\partial v/\partial x$ at z_0. Hence, $\partial u/\partial x$ and $\partial v/\partial x$ exist at z_0 and

$$f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0). \tag{3.3}$$

On the other hand, if $h = it$ where t is real, then $\operatorname{Re}\big((f(z_0+h) - f(z_0))/h\big) = (v(z_0+it) - v(z_0))/t$ and $\operatorname{Im}\big((f(z_0+h) - f(z_0))/h\big) = -(u(z_0+it) - u(z_0))/t$, and so if we let $h \to 0$ along the imaginary axis, then the right-hand side of (3.2) equals $\partial v/\partial y - i\partial u/\partial y$ at z_0. Thus, $\partial v/\partial y$ and $\partial u/\partial y$ exist at z_0 and

$$f'(z_0) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0). \tag{3.4}$$

Identifying real and imaginary parts in (3.3) and (3.4) the Cauchy-Riemann equations (3.1) follow. □

Proposition 3.6. *Let $\Omega \subset \mathbb{C}$ be an open connected set and let f be a holomorphic function in Ω.*

(a) *If $f' = 0$ in Ω, then f is constant.*

(b) *If $|f|$ is constant, then f is constant.*

(c) *If $f(z) \in \mathbb{R}$ for all $z \in \Omega$, then f is constant.*

Parts (b) and (c) say that if the image, $f(\Omega)$, is contained in a circle centered at 0 or in the real axis, then f has to be a constant function. Thus, there are restrictions on what images of holomorphic functions may look like.

Proof of Proposition 3.6. Part (a) follows from Proposition 3.4. Indeed, any two points a and b of Ω may be connected by a piecewise smooth curve $\mathcal{C}$ since Ω is open and connected. Therefore $f(b) - f(a) = \int_{\mathcal{C}} f'(z)dz = 0$ if $f' = 0$, and so $f(a) = f(b)$ for any two points a and b of Ω. It follows that f is constant. A slightly different argument is suggested in Exercise 3.2.

To show part (b) assume that $|f(z)| = c$ for all $z \in \Omega$. If $c = 0$, then $f(z) = 0$ for all $z \in \Omega$ and we are done. Assume tharefore that $c \neq 0$. We have $c^2 = |f(z)|^2 = f(z)\overline{f(z)}$ and so

$$0 = \frac{\partial(f\bar{f})}{\partial\bar{z}} = f\frac{\partial\bar{f}}{\partial\bar{z}} = f\overline{\frac{\partial f}{\partial z}}.$$

Multiplying by $\bar{f}$ and using that $f\bar{f} = c^2$ we then get $0 = c^2\overline{\frac{\partial f}{\partial z}}$. Since $c \neq 0$ we must have $\partial f/\partial z = 0$, i.e., $f' = 0$. It thus follows from part (a) that f is constant.

To show part (c) assume that $f(z)$ is real for all z. Then $f(z) = \overline{f(z)}$ and so $\partial f/\partial z = \partial\bar{f}/\partial z = \overline{(\partial f/\partial\bar{z})} = 0$, see Exercise 2.3. Hence, $f' = 0$ and so f is constant by part (a). □

We conclude this section by saying something about "holomorphicity at ∞". Let f be a holomorphic function defined in $\Omega = \{z \in \mathbb{C};\ |z| > r\}$. If we think of Ω as a subset of the Riemann sphere, then Ω is a "hat" on it with the north pole removed. The Möbius transformation $g(z) = 1/z$ interchanges the north and the south poles and transforms Ω to the set $\tilde{\Omega} = \{w \in \mathbb{C};\ 0 < |w| < 1/r\}$. Then f transforms to a holomorphic function $\tilde{f}$ defined on $\tilde{\Omega}$ by setting $\tilde{f}(w) := f(1/w)$. If it happens that $\tilde{f}$, a priori holomorphic in the punctured disc $\{0 < w < 1/r\}$, can be extended to a holomorphic function in the disc $\{|w| < 1/r\}$, then we say that f is holomorphic at ∞.

3.2 Cauchy's theorem and formula

Without further ado, here is

Theorem 3.7 (Cauchy's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded open set with piecewise smooth positively oriented boundary $\partial\Omega$. If f is a function holomorphic in an open set containing $\overline{\Omega}$, then*

$$\int_{\partial\Omega} f(z)\,dz = 0.$$

Notice that if f happens to be the derivative of a holomorphic function, then Cauchy's theorem follows from Proposition 3.4.

Proof of Cauchy's theorem. From Green's formula in complex notation, see (2.9), we have

$$\int_{\partial\Omega} f(z)\, dz = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar{z}}\, dxdy = 0$$

since f is holomorphic. $\square$

Remark 3.8. As with Green's formula, the result also holds if $\partial\Omega$ is a union of curves $\partial\Omega_j$ provided that they satisfy the hypothesis and that we construe the integral over $\partial\Omega$ as the sum of integrals over the oriented boundary pieces $\partial\Omega_j$.

Cauchy's theorem is fundamental for complex analysis and is the basis of our development of the theory. The main reason why we assume C^1-smoothness in the definition of holomorphicity is that we want to get to Cauchy's theorem quickly and without ad hoc methods.

Our first application of Cauchy's theorem is Cauchy's formula, which provides a way of recovering a holomorphic function in Ω from its values on $\partial\Omega$. In particular, if two holomorphic functions agree on $\partial\Omega$, then they agree on Ω.

Theorem 3.9 (Cauchy's formula). *Let $\Omega \subset \mathbb{C}$ and f be as in Cauchy's theorem. Then*

$$f(a) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)\, dz}{z - a}$$

for each point $a \in \Omega$.

To prove Cauchy's formula we need the following analysis lemma.

Lemma 3.10. *Let φ be a continuous function defined on the disc $\{z \in \mathbb{C};\ |z| < r\}$. Then*

$$\lim_{\varrho \to 0} \int_{|z|=\varrho} \varphi(z) dz/z = 2\pi i \cdot \varphi(0).$$

Proof of Lemma 3.10. In view of Example 2.4 we have

$$\begin{aligned} \int_{|z|=\varrho} \varphi(z) \frac{dz}{z} &= \int_{|z|=\varrho} (\varphi(z) - \varphi(0)) \frac{dz}{z} + \int_{|z|=\varrho} \varphi(0) \frac{dz}{z} \\ &= \int_{|z|=\varrho} (\varphi(z) - \varphi(0)) \frac{dz}{z} + 2\pi i \varphi(0) =: I_\varrho + 2\pi i \varphi(0), \end{aligned}$$

so if we can show that $\lim_{\varrho \to 0} I_\varrho = 0$, then we are done. To this end, let $\epsilon > 0$ be given. Since φ is continuous at 0 there is some $\delta > 0$ such that if $|z| < \delta$ then $|\varphi(z) - \varphi(0)| < \epsilon$. Hence, if $\varrho \leq \delta$ then, by Proposition 2.7,

$$|I_\varrho| = \left| \int_{|z|=\varrho} (\varphi(z) - \varphi(0)) \frac{dz}{z} \right| \leq \sup_{|z|=\varrho} \left| \frac{\varphi(z) - \varphi(0)}{z} \right| \cdot \ell(|z| = \varrho) = \frac{\epsilon}{\varrho} 2\pi\varrho = 2\pi\epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that $\lim_{\varrho \to 0} I_\varrho = 0$. $\square$

Proof of Cauchy's formula. After a translation we may assume that $a = 0$. Since now $0 \in \Omega$ and Ω is open there is some $\varrho > 0$ such that the closure of the disc $D(0, \varrho) = \{z;\ |z| < \varrho\}$ is contained in Ω. Then $\partial\Omega$ together with $\partial D(0, \varrho)$ is the boundary of an open set $\tilde{\Omega} \subset \Omega$. We orient $\partial D(0, \varrho)$ as the boundary of $D(0, \varrho)$ as usual, i.e., $\partial D(0, \varrho)$ is oriented counterclockwise. Letting $-\partial D(0, \varrho)$ denote the curve with opposite orientation we have that $-\partial D(0, \varrho)$ is oriented as a boundary part of $\tilde{\Omega}$. By Cauchy's theorem and Remark 3.8 applied to $\tilde{\Omega}$ and the function $f(z)/z$ we get

$$0 = \int_{\partial\Omega} \frac{f(z) dz}{z} + \int_{-\partial D(0,\varrho)} \frac{f(z) dz}{z} = \int_{\partial\Omega} \frac{f(z) dz}{z} - \int_{|z|=\varrho} \frac{f(z) dz}{z}.$$

But this holds for all sufficiently small $\varrho > 0$, and so, letting $\varrho \to 0$ and using Lemma 3.10, we get the desired equality. $\square$

Exercises

3.1 Let $f(z)$ be holomorphic, let $g(w)$ be holomorphic on the image of f, and let $h = g \circ f$ be the composition. Show, using the chain rule of two-variable real calculus and the Cauchy-Riemann equations (3.1), that $\dfrac{\partial h}{\partial \bar{z}} = 0$ and that $\dfrac{\partial h}{\partial z}(z_0) = \dfrac{\partial g}{\partial w}(f(z_0)) \cdot \dfrac{\partial f}{\partial z}(z_0)$.

3.2 Let f be a holomorphic function in an open connected set and assume that $f' = 0$. Show that $\partial f/\partial x = \partial f/\partial y = 0$ and conclude, using a suitable result from calculus, that f is a constant function.

3.3 Show that if f is holomorphic at ∞ then $\lim_{|z|\to\infty} f(z)$ exists.

3.4 Show that $f(z) = \dfrac{az+b}{cz+d}$ is holomorphic at ∞ if $c \neq 0$.

3.5 Compute the contour integral $\displaystyle\int_{\mathcal{C}} f(z)dz$ when

(a) $f(z) = z^2$ and $\mathcal{C}$ is the line segment from 0 to $1+i$,

(b) $f(z) = \sum_{k=0}^{n} a_k z^k$ and $\mathcal{C}$ is the upper half of the unit circle starting at -1 and ending at 1.

3.6 Let h be holomorphic in a disc D and set $k(z) := \overline{h(z)}$.

(a) Show that k satisfies the Cauchy-Riemann equation(s) at $a \in D$ if and only if $h'(a) = 0$.

(b) Show that k is holomorphic in D if and only if k (and hence also h) is constant.

3.7 Let f be holomorphic in an open connected set Ω and assume that $f(\Omega)$ is contained in a line or in a circle. Show that f is constant.

3.8 Use Cauchy's formula to compute the integral $\displaystyle\int_{\partial D(i,1)} \frac{z^2\,dz}{z^2+1}$.

3.9 Let φ be a C^1-smooth function in $\mathbb{C}$ and assume that $\varphi(z) = 0$ if $|z|$ is sufficiently large. The purpose of this exercise is to outline an argument for the equality

$$\varphi(a) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial\varphi}{\partial\bar{z}}\frac{dxdy}{z-a}. \tag{3.5}$$

(a) Let $\Omega(a,\epsilon) = \mathbb{C} \setminus \overline{D(a,\epsilon)}$. Show that

$$\iint_{\Omega(a,\epsilon)} \frac{\partial\varphi}{\partial\bar{z}}\frac{dxdy}{z-a} = \frac{i}{2}\int_{\partial D(a,\epsilon)} \frac{\varphi(z)\,dz}{z-a}.$$

(Hint: (2.9))

(b) Show (3.5), where the right-hand side is understood as $\displaystyle\lim_{\epsilon\to 0} -\frac{1}{\pi}\iint_{\Omega(a,\epsilon)} \frac{\partial\varphi}{\partial\bar{z}}\frac{dxdy}{z-a}$.

(Hint: Lemma 3.10)

3.10 Let f be holomorphic in the disc $D(a,R)$. Show that f satisfies the *mean value property*, which means that $f(a)$ is the mean value of f on the circle $\partial D(a,r)$ for any $r < R$, that is $f(a) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(a+re^{it})\,dt$ for any $r < R$.

4 Series and power series

Holomorphic functions behave in many ways as polynomials, at least locally. In fact, we will see in Section 4.3 below that any holomorphic function close to any point is given by a power series, which loosely speaking is a “polynomial of infinite degree”. The purpose of Sections 4.1 and 4.2 is to recall some theory of power series to have it at hand when needed.

4.1 Series and convergence tests

Let $\{a_k\}_{k=0}^{\infty}$ be sequence of complex numbers. Then we can define a new sequence $\{s_n\}$ by setting $s_n = \sum_{k=0}^{n} a_k$. This sequence may or may not converge. If it converges we say that the series $\sum a_k$ is *convergent* and we define $\sum_{k=0}^{\infty} a_k := \lim_{n\to\infty} s_n$. If the sequence $\{s_n\}$ is not convergent we say that the series $\sum a_k$ is *divergent*.

From basic properties of limits it follows that if $\sum a_k$ is convergent, then $a_k \to 0$ as $k \to \infty$, and, if $\sum b_k$ is another convergent series and $c \in \mathbb{C}$, then $\sum a_k + cb_k$ is convergent and $\sum_{k=0}^{\infty}(a_k + cb_k) = \sum_{k=0}^{\infty} a_k + c\sum_{k=0}^{\infty} b_k$.

We say that a series $\sum a_k$ is *absolutely convergent* if the series $\sum |a_k|$ is convergent. If $\sum a_k$ is absolutely convergent, then $\sum a_k$ is convergent; this is probably well-known at least if all the a_k are real. An argument showing it in general given the knowledge of it for $a_k \in \mathbb{R}$ is outlined in Exercise 4.1.

We recall the following convergence tests from calculus.

Direct comparison test: Let $\{a_k\}$ and $\{b_k\}$ be sequences of complex and real numbers respectively. Assume that $|a_k| \leq cb_k$ for some constant c and assume that $\sum b_k$ is convergent. Then $\sum a_k$ is absolutely convergent, and in particular convergent.

Ratio test: Let $\{a_k\}$ be a sequence and suppose that $\lim_{k\to\infty} |a_{k+1}|/|a_k|$ exists. If the limit is < 1, then the series $\sum a_k$ is absolutely convergent; if the limit is > 1, then the series $\sum a_k$ is not absolutely convergent (but might still be convergent); if the limit is 1, then we do not get any information about convergence properties of $\sum a_k$.

Root test: Let $\{a_k\}$ be a sequence such that $\lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists. Then the same conclusions as in the Ratio test hold.

Example 4.1 (Geometric series)**.** Let $z \in \mathbb{C}$ and consider the sequence $\{z^k\}_{k=0}^{\infty}$; it is called a geometric sequence. The series $\sum_{k=0}^{\infty} z^k$ is called a geometric series. If $|z| \geq 1$, then $|z^k| \geq 1$ and so z^k cannot go to 0 as $k \to \infty$. Hence, $\sum z_k$ is divergent if $|z| \geq 1$. On the other hand, $(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1}$ so, if $z \neq 1$, we have

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}.$$

If $|z| < 1$, then $\lim_{k\to\infty} z^k = 0$, and we see that the right-hand side goes to $1/(1-z)$ as $k \to \infty$. Hence, $\sum z^k$ is convergent if $|z| < 1$[3] and $\sum_{k=0}^{\infty} z^k = 1/(1 - z)$.

4.2 Functions defined by power series

A *power series* is a series of the form $\sum c_k b^k$, where b and the c_k are complex numbers. Given an $a \in \mathbb{C}$ and a sequence $\{c_k\}$ we get for each $z \in \mathbb{C}$ a power series $\sum c_k(z - a)^k$; it may converge for some values of z (it converges at least for $z = a$) and diverge for other, depending on the sequence $\{c_k\}$. If it converges for z in some set $D \subset \mathbb{C}$, then we can define a function on D by setting $f(z) = \sum_{k=0}^{\infty} c_k(z - a)^k$.

[3] One can also use the Ratio or Root test to check this.

Definition 4.2 (Radius of convergence)**.** The *radius of convergence* of the power series $\sum c_k(z-a)^k$ is defined as $\sup\{|z-a|; \sum c_k(z-a)^k$ is absolutely convergent$\}$.

Let R be the radius of convergence of $\sum c_k(z-a)^k$. Then, by definition, for any $r < R$ there is some w such that $r < |w-a|$ and $\sum c_k(w-a)^k$ is absolutely convergent. Moreover, if $|w-a| > R$, then $\sum c_k(w-a)^k$ cannot be absolutely convergent. Notice that R may be ∞; this is the case if and only if there is, for any $r > 0$, some $w \in \mathbb{C}$ such that $|w-a| > r$ and $\sum c_k(w-a)^k$ is absolutely convergent. Notice also that $R = 0$ if and only if $\sum c_k(z-a)^k$ is absolutely convergent only for $z = a$.

Example 4.3. Compute the radius of convergence of $\sum_k k^2 z^k$.
Solution: We have

$$\lim_{k\to\infty} \frac{|(k+1)^2 z^{k+1}|}{|k^2 z^k|} = \lim_{k\to\infty} (1+1/k)^2|z| = |z|.$$

If $|z| < 1$ it thus follows from the Ratio test that $\sum_k k^2 z^k$ is absolutely convergent. On the other hand, if $|z| \geq 1$, then $\sum_k k^2 z^k$ is divergent since $k^2 z^k$ does not go to 0 as $k \to \infty$. Hence the radius of convergence is 1.

We see in this example that the series converges absolutely for all z in the disc determined by the radius of convergence, and diverges for all z outside of that disc. This is not a coincidence, in fact we have

Lemma 4.4. *Let $\sum_k c_k(z-a)^k$ be a power series with radius of convergence R.*

(a) *If $|z-a| < R$ then $\sum_k c_k(z-a)^k$ is absolutely convergent.*

(b) *If $|z-a| > R$ then $\sum_k c_k(z-a)^k$ is divergent.*

Proof. After a translation we may assume that $a = 0$. To prove part (a), let z be a point such that $|z| < R$. By definition of R there is a w such that $|z| < |w| < R$ and $\sum c_k w^k$ is absolutely convergent. Then, since $|z/w| < 1$, we have

$$|c_k z^k| \leq |c_k w^k| \left|\frac{z}{w}\right|^k \leq |c_k w^k|,$$

and so it follows by the Direct comparison test that $\sum |c_k z^k|$ is convergent, i.e., $\sum c_k z^k$ is absolutely convergent.

To prove part (b), let z be a point such that $|z| > R$. By definition of R we know that $\sum |c_k z^k|$ is divergent but we want to show that $\sum c_k z^k$ is divergent. Assume therefore, to get a contradiction, that $\sum c_k z^k$ is convergent. Then $|c_k z^k| \to 0$ as $k \to \infty$ and in particular the sequence $\{c_k z^k\}_k$ has to be bounded. There is therefore an $M > 0$ such that $|c_k z^k| \leq M$ for all k. Let w be any point such that $R < |w| < |z|$. Then

$$|c_k w^k| = |c_k z^k| \left|\frac{w}{z}\right|^k \leq M \left|\frac{w}{z}\right|^k,$$

and, since $|w/z| < 1$, the geometric series $\sum M|w/z|^k$ converges. Thus, by the Direct comparison test, $\sum |c_k w^k|$ converges. But since $|w| > R$ this contradicts the definition of R. Our assumption must therefore be wrong and hence $\sum c_k z^k$ cannot be convergent. $\square$

From this lemma we thus see that if the power series $\sum c_k(z-a)^k$ has radius of convergence R then we get a function f defined in the disc $D(a,R) = \{z \in \mathbb{C}; |z-a| < R\}$ by setting $f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k$. We will show in the remainder of this section that functions defined by power series in fact are C^∞-smooth and holomorphic. We begin with the preliminary

Lemma 4.5. *Let $\sum c_k z^k$ be a power series with radius of convergence R. Then*

(a) $\sum c_k k z^{k-1}$ *has radius of convergence R,*

(b) $\sum_k c_k k(k-1)\cdots(k-\ell)z^{k-\ell-1}$ *has radius of convergence R for for each natural number ℓ.*

Proof. Notice that part (b) follows from repeated use of part (a).

To prove part (a) we first show that $\sum c_k k z^{k-1}$ has radius of convergence $\le R$. If not, then there is a z such that $|z| > R$ and $\sum c_k k z^{k-1}$ is absolutely convergent. Since $|c_k z^k| = |z||c_k z^{k-1}| \le |z||c_k k z^{k-1}|$ it thus follows from the Direct comparison test that $\sum |c_k z^k|$ is convergent, contradicting the definition of R since $|z| > R$.

To show that $\sum c_k k z^{k-1}$ has radius of convergence $\ge R$, let z be any point $\neq 0$ such that $|z| < R$. From the definition of R there is a w such that $|z| < |w| < R$ and $\sum |c_k w^k|$ is convergent. Since $|z/w| < 1$ we know from one-variable calculus that $k|z/w| \to 0$ as $k \to \infty$, and so, in particular, there is a constant $M > 0$ such that $k|z/w| \le M$ for all k. Hence,

$$|c_k k z^{k-1}| = \frac{k}{|z|}\left|\frac{z}{w}\right|^k |c_k w^k| \le \frac{M}{|z|}|c_k w^k|$$

and the Direct comparison test implies that $\sum c_k k z^{k-1}$ is absolutely convergent. □

Theorem 4.6. *Let $\sum c_k (z-a)^k$ be a power series with radius of convergence $R > 0$ and define $f\colon D(a,R) \to \mathbb{C}$ by $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$. Then*

(a) *f is C^∞-smooth in $D(a,R)$,*

(b) *f is holomorphic in $D(a,R)$,*

(c) *for each natural number ℓ, $f^{(\ell)}(z) = \sum_{k=\ell}^{\infty} c_k k(k-1)\cdots(k-\ell+1)(z-a)^{k-\ell}$.*

Notice that part (c) says that the ℓ^{th} derivative of f is given by the term-wise differentiated power series, which by by Lemma 4.5 has radius of convergence R.

Proof of Theorem 4.6. After a translation we may assume that $a = 0$. We begin with the technical part of the proof which is to show that for each fixed $z \in D(0,R)$ we have

$$f(z+h) - f(z) = h\sum_{k=1}^{\infty} c_k k z^{k-1} + O(|h|^2). \tag{4.1}$$

By the Binomial theorem we have

$$\begin{aligned}
f(z+h)-f(z) &= \sum_{k=1}^{\infty} c_k\big((z+h)^k - z^k\big)\\
&= \sum_{k=1}^{\infty} c_k \sum_{\ell=1}^{k}\binom{k}{\ell} z^{k-\ell}h^{\ell}\\
&= h\sum_{k=1}^{\infty} c_k \sum_{\ell=1}^{k}\binom{k}{\ell} z^{k-\ell}h^{\ell-1}\\
&= h\sum_{k=1}^{\infty} c_k \left(\binom{k}{1} z^{k-1} + h\sum_{\ell=2}^{k}\binom{k}{\ell} z^{k-\ell}h^{\ell-2}\right)\\
&= h\sum_{k=1}^{\infty} c_k k z^{k-1} + h^2\sum_{k=2}^{\infty} c_k \sum_{\ell=2}^{k}\binom{k}{\ell} z^{k-\ell}h^{\ell-2}\\
&= h\sum_{k=1}^{\infty} c_k k z^{k-1} + h^2 B(h).
\end{aligned}$$

To show (4.1) we thus need to show that $B(h)$ is bounded as $h \to 0$, i.e., that there are numbers C and $\delta > 0$ such that $|B(h)| \le C$ when $|h| \le \delta$. Take $\delta < R - |z|$. Then, if $|h| \le \delta$, we have

$$\begin{aligned}
|B(h)| &= \left|\sum_{k=2}^{\infty} c_k \sum_{\ell=2}^{k}\binom{k}{\ell} z^{k-\ell}h^{\ell-2}\right|\\
&\le \sum_{k=2}^{\infty} |c_k| \sum_{m=0}^{k-2}\binom{k}{m+2} |z|^{k-2-m}|h|^{m}\\
&= \sum_{k=2}^{\infty} |c_k| \sum_{m=0}^{k-2}\frac{k(k-1)}{(m+2)(m+1)}\binom{k-2}{m} |z|^{k-2-m}|h|^{m}\\
&\le \sum_{k=2}^{\infty} |c_k| k(k-1) \sum_{m=0}^{k-2}\binom{k-2}{m} |z|^{k-2-m}|h|^{m}\\
&= \sum_{k=2}^{\infty} |c_k| k(k-1)\big(|z|+|h|\big)^{k-2}\\
&\le \sum_{k=2}^{\infty} |c_k| k(k-1)\big(|z|+\delta\big)^{k-2},
\end{aligned}$$

where we in the third step have used basic properties of binomial coefficients and in the fifth step have used the Binomial theorem backwards. Moreover, in view of Lemma 4.5, the last series is convergent since $|z| + \delta < R$. Hence, we may take the sum of this series as the constant C, and so (4.1) follows. We notice in particular that it by (4.1) follows that f is continuous in $D(0, R)$.

From (4.1) it also readily follows that f is complex differentiable at all points $z \in D(0, R)$ with derivative $f'(z)$ given by $\sum_{k=1}^{\infty} c_k k z^{k-1}$, which has radius of convergence R by Lemma 4.5. By Theorem 3.5 the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in $D(0, R)$ and $f' = \partial f/\partial x = -i\partial f/\partial y$, see (3.3) and (3.4). Hence, $\partial f/\partial x$ and $\partial f/\partial y$ are given by power series converging in $D(0, R)$ and so, as in the first part of the proof, $\partial f/\partial x$ and $\partial f/\partial y$ are continuous in $D(0, R)$.

Thus, f is C^1-smooth and complex differentiable in $D(0,R)$, i.e., f is holomorphic in $D(0,R)$, and, moreover, $f'(z)=\sum_{k=1}^{\infty} c_k k z^{k-1}$. But then we can apply precisely the same reasoning to f' to see that f' is holomorphic in $D(0,R)$ and $f''(z)=\sum_{k=2}^{\infty} c_k k(k-1)z^{k-2}$, which again by Lemma 4.5 has radius of convergence R. Thus f'' is holomorphic in $D(0,R)$ with derivative given by a power series converging in $D(0,R)$. Continuing in this way (formally using induction) Theorem 4.6 follows. □

4.3 Holomorphic functions are power series locally

We will see that if f is a holomorphic function in an open set $\Omega\subset\mathbb{C}$, then for any given $a\in\Omega$ and any disc $D(a,r)$ centered at a such that $\overline{D(a,r)}\subset\Omega$, $f\restriction_{D(a,r)}$ is given by a power series $\sum c_k(z-a)^k$ converging in $D(a,r)$. This is the content of the next result which is a version of Taylor's theorem for holomorphic functions.

Theorem 4.7. *Let f be a holomorphic function in an open set containing $\overline{D(a,r)}$. Then there is a unique power series $\sum c_k(z-a)^k$ converging absolutely in $D(a,r)$ such that $f(z)=\sum_{k=0}^{\infty} c_k(z-a)^k$ for all $z\in D(a,r)$. Moreover, the coefficients c_k are given by*

$$c_k=\frac{1}{2\pi i}\int_{\partial D(a,r)}\frac{f(w)\,dw}{(w-a)^{k+1}}=\frac{f^{(k)}(a)}{k!}. \tag{4.2}$$

Remark 4.8. By Cauchy's theorem, the integration contour $\partial D(a,r)$ in (4.2) may be replaced by any piecewise smooth simple closed curve enclosing a.

Remark 4.9. If f is holomorphic merely in $D(a,r)$, then the conclusions of Theorem 4.7 still hold except for that we need to replace the integration contour $\partial D(a,r)$ in (4.2) by a slightly smaller curve. To see this we simply apply Theorem 4.7 to the smaller disc $D(a,r-\epsilon)$ and let $\epsilon\to 0$.

Since holomorphicity and smoothness are local properties and since convergent power series define holomorphic C^∞-smooth functions by Theorem 4.6 we get the following corollary of Theorem 4.7.

Corollary 4.10. *If f is holomorphic in an open set $\Omega\subset\mathbb{C}$, then $f\in C^\infty(\Omega)$ and $f^{(k)}$ is holomorphic in Ω for any natural number k.*

Proof of Theorem 4.7. After translation and scaling we may assume that $a=0$ and $r=1$.

First, notice that if $\sum c_k z^k$ is absolutely convergent in $D(0,1)$ and $f(z)=\sum_{k=0}^{\infty} c_k z^k$ there, then $f(0)=c_0$ and, by differentiating term-wise ℓ times,

$$f^{(\ell)}(0)=\sum_{k=\ell}^{\infty} c_k\,\ell!\,z^{k-\ell}\Big|_{z=0}=c_\ell \ell!.$$

Hence, any absolutely convergent series in $D(0,1)$ representing f there must have coefficients $c_k=f^{(k)}(0)/k!$. This proves the uniqueness part.

Now we define coefficients c_k by $c_k=(2\pi i)^{-1}\int_{|w|=1} f(w)dw/w^{k+1}$, i.e., so that the left-hand equality of (4.2) holds, and we set

$$A_N(z):=f(z)-\sum_{k=0}^{N} c_k z^k.$$

To prove the theorem it suffices to show that $\lim_{N\to\infty} A_N(z) = 0$ for each fixed $z \in D(0,1)$ because then $\sum c_k z^k$ has to be convergent for each $z \in D(0,1)$ with sum $f(z)$. Moreover, $\sum c_k z^k$ then also has to be absolutely convergent in $D(0,1)$ since if not, then $\sum c_k z^k$ would not be convergent in $D(0,1)$ by Lemma 4.4. From the uniqueness part it will then also follow that the right-hand equality of (4.2) holds.

To show that $\lim_{N\to\infty} A_N(z) = 0$ we plug in the definition of the coefficients c_k, use Cauchy's formula to write $f(z) = (2\pi i)^{-1} \int_{|w|=1} f(w)\, dw/(w-z)$, and compute:

$$
\begin{aligned}
A_N(z) &= \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)\,dw}{w-z} - \sum_{k=0}^{N} z^k \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)\,dw}{w^{k+1}} \\
&= \frac{1}{2\pi i}\int_{|w|=1} \left(\frac{f(w)}{w-z} - \sum_{k=0}^{N} \frac{f(w)}{w}\left(\frac{z}{w}\right)^k \right) dw \\
&= \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w}\left(\frac{1}{1-z/w} - \sum_{k=0}^{N} (z/w)^k \right) dw \\
&= \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w}\left(\frac{1}{1-z/w} - \frac{1-(z/w)^{N+1}}{1-z/w} \right) dw \\
&= \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w}\,\frac{(z/w)^{N+1}}{1-z/w}\, dw,
\end{aligned}
$$

where we in the forth equality have used the formula for a geometric sum, cf. Example 4.1, noticing that $|z/w| < 1$ for any $z \in D(0,1)$ and any w with $|w| = 1$. Now, f is in particular continuous on the circle $\{|w| = 1\}$ and so $|f(w)| \le M$ if $|w| = 1$ for some constant M. Moreover, by the Reverse triangle inequality, $|1 - z/w| \ge 1 - |z/w| = 1 - |z|$ if $|w| = 1$, and hence

$$
\sup_{|w|=1} \left| \frac{f(w)}{w}\,\frac{(z/w)^{N+1}}{1-z/w} \right| = \sup_{|w|=1} |f(w)| \frac{|z|^{N+1}}{|1-z/w|} \le M \frac{|z|^{N+1}}{1-|z|}.
$$

By Proposition 2.7 we thus get

$$
|A_N(z)| = \frac{1}{2\pi} \left| \int_{|w|=1} \frac{f(w)}{w}\,\frac{(z/w)^{N+1}}{1-z/w}\, dw \right| \le \frac{M}{2\pi}\frac{|z|^{N+1}}{1-|z|} \ell\big(|w|=1\big) = M \frac{|z|^{N+1}}{1-|z|},
$$

and since $|z| < 1$ we see that $\lim_{N\to\infty} |A_N(z)| = 0$. □

Exercises

4.1 Let $a_k = x_k + i y_k$ be a sequence of complex numbers such that $\sum |a_k|$ is convergent. Show that $\sum_k |x_k|$ and $\sum_k |y_k|$ are convergent. Use Lemma 14.10 to conclude that $\sum a_k$ is convergent.

4.2 Compute the radius of convergence of the power series

$$
\text{(a)}\ \sum_k z^k/k^3, \quad \text{(b)}\ \sum_k (-1)^k k^5 z^k, \quad \text{(c)}\ \sum_k 8^k z^{3k}, \quad \text{(d)}\ \sum_k z^k/k^k, \quad \text{(e)}\ \sum_k k! z^k.
$$

4.3 Find a power series representing f in the disc D where

(a) $f(z) = 1/(z+1)$ and $D = D(0,1)$,

(b) $f(z) = 1/z$ and $D = D(i, 1)$,

(c) $f(z) = 1/(z^2 - z)$ and $D = D(1/2, 1/2)$,

(d) $f(z) = 1/(z^2 + z + 1)$ and $D = D(0, 1)$.

4.4 Find a power series representing f in $D(0, 1)$ where

$$\text{(a)}\ f(z) = \frac{1}{(1-z)^2}, \quad \text{(b)}\ f(z) = \frac{1}{(1-z)^3}.$$

4.5 *Cauchy's estimate.* Show that if f is holomorphic in $D(0, r)$ and $|f(z)| \leq M$ for all $z \in D(0, r)$, then $|f^{(k)}(0)| \leq k!M/r^k$ for all integers $k \geq 0$. (Hint: Use Theorem 4.7 and Proposition 2.7.)

5 Liouville's theorem and the Fundamental theorem of algebra

We show two more important consequences of Cauchy's formula.

5.1 Liouville's theorem

Liouville's theorem says that a non-constant holomorphic function in $\mathbb{C}$ cannot be bounded. Put in another way, if f is a holomorphic function in $\mathbb{C}$ such that its image $f(\mathbb{C})$ is contained in some disc, then f is constant.

Theorem 5.1 (Liouville's theorem)**.** *Let f be a holomorphic function in $\mathbb{C}$ and assume that there is an $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is a constant function.*

Proof. Let $a, b \in \mathbb{C}$ be arbitrary points. It suffices to show that $f(a) = f(b)$. Let $R > 2\max\{|a|, |b|\}$. Then, for any z with $|z| = R$ we have

$$|z - a| > R/2 \quad \text{and} \quad |z - b| > R/2.$$

By Cauchy's formula we also have

$$f(a) - f(b) = \frac{1}{2\pi i}\int_{\partial D(0,R)} \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i}\int_{\partial D(0,R)} \frac{f(z)\,dz}{z-b} = \frac{1}{2\pi i}\int_{\partial D(0,R)} \frac{f(z)(a-b)\,dz}{(z-a)(z-b)}.$$

Hence, by Proposition 2.7 we get

$$\begin{aligned} \big|f(a) - f(b)\big| &= \frac{1}{2\pi}\left|\int_{\partial D(0,R)} \frac{f(z)(a-b)\,dz}{(z-a)(z-b)}\right| \\ &\leq \frac{1}{2\pi}\sup_{|z|=R} \frac{|f(z)||a-b|}{|z-a||z-b|} \cdot 2\pi R \\ &\leq \frac{M|a-b|R}{(R/2)(R/2)} = \frac{4M|a-b|}{R}. \end{aligned}$$

But this holds for all sufficiently large R and since $4M|a-b|/R \to 0$ as $R \to \infty$ it follows that $f(a) = f(b)$. □

A slightly different proof, giving a more general Liouville theorem, is outlined in Exercise 5.3.

5.2 The fundamental theorem of algebra

The fundamental theorem of algebra says that any polynomial equation $p(z) = 0$ has a complex solution (unless p is a non-zero constant). Let the degree of p be $n > 0$ and let $a_1 \in \mathbb{C}$ be a solution of $p(z) = 0$. By polynomial division it then follows that $p(z) = p_1(z)(z - a_1)$, where p_1 is a polynomial of degree $n-1$. Applying the Fundamental theorem of algebra to p_1 we see that there is an $a_2 \in \mathbb{C}$ such that $p_1(a_2) = 0$ and then, by polynomial division, $p_1(z) = p_2(z)(z-a_2)$ for some polynomial p_2 of degree $n-2$. Continuing in this way we finally get a polynomial p_n of degree 0, i.e., p_n is a constant c, such that $p_{n-1}(z) = c(z - a_n)$. Hence, the Fundamental theorem of algebra implies that any polynomial of degree n with complex coefficients can be factorized as

$$p(z) = c(z - a_1) \cdot (z - a_2) \cdots (z - a_n),$$

where c and the a_j's are complex numbers. The terminology to express this property of the complex numbers is to say that $\mathbb{C}$ is *algebraically closed*.

Theorem 5.2 (Fundamental theorem of algebra)**.** *Let $p(z)$ be a non-constant polynomial with complex coefficients. Then there is an $a \in \mathbb{C}$ such that $p(a) = 0$.*

Proof. Assume, to get a contradiction, that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ is a holomorphic function in $\mathbb{C}$. Since p is a non-constant polynomial, $|p(z)| \to \infty$ as $|z| \to \infty$; an argument for this fact is outlined in Exercise 5.1. It follows that there is an $M > 0$ such that $1/|p(z)| \leq M$ for all $z \in \mathbb{C}$. In fact, since $|p(z)| \to \infty$ as $|z| \to \infty$ there is an $R_0 > 0$ such that $1/|p(z)| \leq 1$ if $|z| > R_0$ and, moreover, since $1/|p(z)|$ is a continuous function on the closed and bounded set $\overline{D(0, R_0)}$ there is an M' such that $1/|p(z)| \leq M'$ if $z \in \overline{D(0, R_0)}$. Hence, we may take $M = \max\{1, M'\}$. Now, since $1/p(z)$ is a bounded holomorphic function in $\mathbb{C}$ Liouville's theorem shows that $1/p(z)$ is constant, but then $p(z)$ is constant, which is a contradiction. Our assumption must thus be false and so there is an $a \in \mathbb{C}$ such that $p(a) = 0$. □

Exercises

5.1 Let $p(z) = \sum_{k=0}^{n} c_k z^k$ where $c_n \neq 0$. Show that $\lim_{|z| \to \infty} \dfrac{|p(z)|}{|z|^n} = |c_n|$ and conclude that $|p(z)| \geq (1/2)|c_n||z|^n$ if $|z|$ is sufficiently large.

5.2 Show that if f is holomorphic in $\mathbb{C}$ and $|f(z)| \geq M > 0$ for all $z \in \mathbb{C}$, then f is constant.

5.3 *Generalized Liouville theorem.* The purpose of this exercise is to show that if f is a holomorphic function in $\mathbb{C}$ and $|f(z)| \leq C(1 + |z|)^n$ for some constants C and n and all $z \in \mathbb{C}$, then f is a polynomial of degree $\leq n$. This means roughly speaking that if $|f(z)|$ does not grow faster than a polynomial as $|z| \to \infty$, then f is a polynomial.

(a) Use Exercise 4.5 to show that $|f^{(k)}(0)| \leq k!C(1+r)^n/r^k$ for all $r > 0$ and all integers $k \geq 0$.

(b) Use Theorem 4.7 to conclude that f is a polynomial of degree $\leq n$.

6 Elementary functions via power series

We will use power series to define exponential, trigonometric, and hyperbolic functions. We then define logarithms by inverting the exponential function. As the exponential function turns out not to be injective this requires some care and involves making choices.

6.1 Exponential functions

The series $\sum z^k/k!$ converges absolutely in $\mathbb{C}$; this follows for instance from the Ratio test since $|z^{k+1}/(k+1)!|/|z^k/k!| = |z|/(k+1) \to 0$ as $k \to \infty$ for any $z \in \mathbb{C}$. Thus, it defines a holomorphic function in $\mathbb{C}$ that we take as definition of the exponential function, i.e.,

$$e^z := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{6.1}$$

Notice that we now have two potentially different definitions of $e^{i\theta}$ for $\theta \in \mathbb{R}$ but we will soon see that they coincide. We notice also that (6.1) is an extension to $\mathbb{C}$ of the exponential function on $\mathbb{R}$, i.e., that the right-hand side of (6.1), with $z = x \in \mathbb{R}$, equals e^x as defined in elementary calculus. Indeed, from calculus we know that $\sum_{k=0}^{\infty} x^k/k!$ is the Taylor series of e^x.

The following result extends familiar computation rules of the exponential function from $\mathbb{R}$ to $\mathbb{C}$.

Theorem 6.1. *The exponential function e^z has the following properties:*

$$\text{(a)}\ e^0 = 1, \quad \text{(b)}\ \frac{\partial e^z}{\partial z} = e^z, \quad \text{(c)}\ e^{z+w} = e^z e^w, \quad \text{(d)}\ e^z \neq 0\ \ \forall z \in \mathbb{C}.$$

Proof. Part (a) is clear from the definition (6.1). Part (b) follows since

$$\frac{\partial e^z}{\partial z} = \frac{\partial}{\partial z} \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{\partial}{\partial z} \frac{z^k}{k!} = \sum_{k=1}^{\infty} \frac{k z^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!} = e^z,$$

where we have used Theorem 4.6 (c) to differentiate the series term-wise.

To show part (c), let $a \in \mathbb{C}$ be an arbitrary complex number and consider the function $f(z) = e^z e^{a-z}$; notice that e^{a-z} is a composition of holomorphic functions and thus holomorphic. Differentiating, cf. Exercise 2.1 (b) and Exercise 3.1, and using part (a) we get

$$f'(z) = e^z e^{a-z} - e^z e^{a-z} = 0.$$

Hence, by Proposition 3.6, f is constant and so $f(z) = f(0) = e^a$ in view of part (a). Thus, $e^z e^{a-z} = e^a$ for all $a, z \in \mathbb{C}$ and choosing $a = z + w$ we get $e^z e^w = e^{z+w}$.

Part (d) follows from parts (a) and (c) since for any $z \in \mathbb{C}$, $e^z e^{-z} = e^0 = 1$, and so e^z cannot be 0. □

Another property of the exponential function worth pointing out is the following.

Proposition 6.2. *If $z = x + iy$, then $|e^z| = e^x$. In particular, $|e^{iy}| = 1$.*

Proof. In view of Exercise 6.1 we have

$$|e^z|^2 = e^z \overline{e^z} = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2x} = (e^x)^2.$$

□

We define the exponential function with base $a \in (0, \infty)$ by setting $a^z := e^{z \cdot \log a}$. It is clear that the standard computation rules of these exponential functions then extend to $\mathbb{C}$ too.

6.2 Trigonometric and hyperbolic functions

We define the basic trigonometric and hyperbolic functions by the following series.

$$\cos z := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \tag{6.2}$$

$$\sin z := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \tag{6.3}$$

$$\cosh z := 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \tag{6.4}$$

$$\sinh z := z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \tag{6.5}$$

The radius of convergence of these series is ∞, Exercise 6.2, and, as with the exponential function, we see that these definitions extend the usual trigonometric and hyperbolic functions from $\mathbb{R}$ to holomorphic functions in $\mathbb{C}$. Moreover, by differentiating term-wise (do it!) we get that

$$\frac{\partial \cos z}{\partial z} = -\sin z, \qquad \frac{\partial \sin z}{\partial z} = \cos z,$$
$$\frac{\partial \cosh z}{\partial z} = \sinh z, \qquad \frac{\partial \sinh z}{\partial z} = \cosh z.$$

We notice also that $\cos z = \cosh(iz)$, since $(iz)^{2k} = (-1)^k z^{2k}$, and that $i \sin z = \sinh(iz)$ since $(iz)^{2k+1} = i(-1)^k z^{2k+1}$. By summing the series (6.4) and (6.5) with z replaced by iz we thus obtain

$$\cos z + i \sin z = \cosh(iz) + \sinh(iz) = e^{iz}.$$

In particular, $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$.

The standard addition formulas for sin and cos extend to hold also for complex variables. This can be checked, Exercise 6.4, using, e.g., the following identities, which follow easily from the definitions.

$$\cos z = \frac{1}{2}\big(e^{iz} + e^{-iz}\big), \qquad \sin z = \frac{1}{2i}\big(e^{iz} - e^{-iz}\big), \tag{6.6}$$

$$\cosh z = \frac{1}{2}\big(e^{z} + e^{-z}\big), \qquad \sinh z = \frac{1}{2}\big(e^{z} - e^{-z}\big). \tag{6.7}$$

As we know, $|\cos x| \leq 1$ and $|\sin x| \leq 1$ for $x \in \mathbb{R}$ but this is not the case for a complex variable. In fact, both cos and sin are unbounded functions on $\mathbb{C}$. For instance, by (6.6),

$$|\cos(x+iy)| = \frac{1}{2}\big|e^{ix-y} + e^{-ix+y}\big| \geq \frac{1}{2}\big||e^{ix-y}| - |e^{-ix+y}|\big| = \frac{1}{2}\big|e^{-y} - e^{y}\big|,$$

which goes to $+\infty$ as y goes to either $+\infty$ or $-\infty$.

Another thing we need to get used to is that the exponential function is periodic in the imaginary direction since $e^{z+2\pi ik} = e^z e^{2\pi ik} = e^z(\cos 2\pi k + i \sin 2\pi k) = e^z$ for any integer k. In particular, the exponential function is not injective, which causes problems when we will try to define logarithms in the next section.

6.3 Logarithms, arguments, and general powers

As mentioned, the exponential function is not injective on $\mathbb{C}$ and therefore it cannot have an inverse defined on all of $\mathbb{C}$. We encounter a similar problem in calculus when defining arccos and arcsin, and the way out is to restrict the domains of definition of cos and sin to make them injective. We will do the same thing with the exponential function.

Consider the horizontal strip $S_\varphi = \{z \in \mathbb{C};\ \varphi < \operatorname{Im} z \le \varphi + 2\pi\}$. Then the restriction of e^z to S_φ is injective. To see this, assume that $e^z = e^w$. Then $e^{z-w} = 1$ and so, by Exercise 6.3, $z - w = 2\pi i k$ for some integer k. If both z and w are in S_φ, then their imaginary parts cannot differ by a non-zero integer multiple of 2π and hence $k = 0$, i.e., $z = w$. Thus, we may invert the restriction of e^z to S_φ. This amounts to finding $w \in S_\varphi$ such that $e^w = z$ given some z. To do this we write z on polar form;

$$z = |z|e^{i\theta} = e^{\log|z|}e^{i\theta} = e^{\log|z| + i\theta}, \tag{6.8}$$

where θ is an appropriate angle. Thus we may take $w = \log|z| + i\theta$. As above we notice that there is no unique θ such that (6.8) holds; θ is only determined up to an integer multiple of 2π, i.e., two different θ's doing the job may differ by an integer multiple of 2π. This is "the reason" why the exponential function is not injective. But since we require w to be in the strip S_φ, θ must satisfy $\varphi < \theta \le \varphi + 2\pi$ and then there is no ambiguity and so θ is uniquely determined.

Given $\varphi \in \mathbb{R}$ we define the argument function $\arg_\varphi$ by setting $\arg_\varphi z = \theta$, where $\varphi < \theta \le \varphi + 2\pi$ and $z = |z|e^{i\theta}$. Notice that $\arg_\varphi$ is not defined at 0. Notice also that $\arg_\varphi$ is not continuous on $\mathbb{C} \setminus \{0\}$ because if it were, then the the restriction of $\arg_\varphi$ to the unit circle would be continuous, and then $\psi(t) := \arg_\varphi(e^{it})$ would be continuous on $\mathbb{R}$. But ψ jumps up 2π at the points $t = \varphi + 2\pi k$, $k \in \mathbb{Z}$. On the other hand, $\arg_\varphi$ restricted to the cut plane $\mathbb{C}_\varphi := \mathbb{C} \setminus \{re^{i\varphi} \in \mathbb{C};\ r \ge 0\}$, i.e., points on the ray starting at 0 and going through $e^{i\varphi}$ are not allowed, is smooth (Exercise 6.5). We define the logarithm $\log_\varphi$[4] on $\mathbb{C}_\varphi$ by

$$\log_\varphi z = \log|z| + i \arg_\varphi z.$$

Then $\log_\varphi$ is smooth and $u(z) = \log|z|$ and $v(z) = \arg_\varphi$ satisfy the Cauchy-Riemann equations (3.1) in $\mathbb{C}_\varphi$, Exercise 6.6. Hence, $\log_\varphi$ is holomorphic in $\mathbb{C}_\varphi$. Knowing this we can compute the derivative of $\log_\varphi$ using the Chain rule. In fact, for any $z \in \mathbb{C}_\varphi$ we have $z = e^{\log_\varphi z}$, and differentiating this identity we get $1 = e^{\log_\varphi z}(\log_\varphi z)' = z(\log_\varphi z)'$. Hence,

$$(\log_\varphi z)' = \frac{1}{z}.$$

Different choices of φ give different logarithms and one often says that one chooses branch of the logarithm. The choice $\varphi = -\pi$ is called the *principal branch* and we will simply write Log to denote it. Similarly, we let $\operatorname{Arg} := \arg_{-\pi}$ be the principal branch of the argument functions.

Let $\alpha \in \mathbb{R}$. Then, given φ, we define $z^\alpha := e^{\alpha(\log_\varphi z)}$. Notice that the left-hand side depends in general on the choice of φ even though it is not visible from the notation. When dealing with general powers it is necessary to take care to specify what one means! Notice also that if α is an integer, then there is no ambiguity.

Example 6.3. Compute all possible values of $(2i)^{1/2}$.

Solution: We shall compute $e^{\left(\log_\varphi(2i)\right)/2}$ for all possible values of φ. The possible values of $\arg_\varphi 2i$ are $\pi/2 + 2\pi k$, $k \in \mathbb{Z}$, so the possible values of $\log_\varphi(2i)$ are $\log_\varphi(2i) = \log 2 + i(\pi/2 + 2\pi k)$

[4] This should not be confused with logarithms with respect to other bases than e; we are not concerned with such logarithms here.

for $k \in \mathbb{Z}$. Thus, the possible values of $(2i)^{1/2}$ are

$$\begin{aligned} e^{\left(\log 2+i(\pi/2+2\pi k)\right)/2} &= e^{(\log 2)/2}e^{i(\pi/4+\pi k)} = \sqrt{2}e^{i\pi/4}e^{i\pi k} \\ &= \sqrt{2}\frac{1+i}{\sqrt{2}}e^{i\pi k} = \begin{cases} 1+i, & \text{if } k \text{ is odd,} \\ -1-i, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

We notice in particular that the possible values of $(2i)^{1/2}$ are the solutions of the equation $z^2 = 2i$. In general, if $w \neq 0$, then the possible values of $w^{1/n}$ are the solutions of the equation $z^n = w$.

Exercises

6.1 Let $f(z) = \sum_k c_k z^k$, where the series has radius of convergence R. Show that $\overline{f(z)} = \sum_k \bar{c}_k \bar{z}^k$ for all z with $|z| < R$.

6.2 Show that the series in (6.2), (6.3), (6.4), and (6.4) have infinite radius of convergence.

6.3 Show that

(a) $e^z = 1$ if and only if $z = 2\pi i k$, $k \in \mathbb{Z}$,

(b) $\cos z = 0$ if and only if $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$,

(c) $\sin z = 0$ if and only if $z = k\pi$, $k \in \mathbb{Z}$.

6.4 Show that

(a) $\cos(z+w) = \cos z \cos w - \sin z \sin w$ for all $z, w \in \mathbb{C}$,

(b) $\sin(z+w) = \sin z \cos w + \sin w \cos z$ for all $z, w \in \mathbb{C}$.

6.5 Show that $\arg_\varphi$ is smooth on the cut plane $\mathbb{C}_\varphi$ and compute the partial derivatives $\partial \arg_\varphi / \partial x$ and $\partial \arg_\varphi / \partial y$.

6.6 Let $u(x,y) = \log\sqrt{x^2+y^2}$ and $v(x,y) = \arg_\varphi(x+iy)$. Show that u and v satisfy the Cauchy-Riemann equations (3.1) in $\mathbb{C}_\varphi$.

6.7 Let f be holomorphic in $\mathbb{C}$. Show that f is constant if

(a) $e^{f(z)}$ is bounded,

(b) $\operatorname{Re} f$ or $\operatorname{Im} f$ is bounded. (Hint: Part (a) and Proposition 6.2.)

6.8 Write e^z as a power series $\sum_k c_k (z-1)^k$ in $\mathbb{C}$.

6.9 Write $\sin^2 z$ as a power series $\sum_k c_k z^k$ in $\mathbb{C}$.

7 Conformal mappings

Concrete problems often become computationally easier if the underlying geometry is simple. For instance, certain physical problems amount to solving a differential equation in an open set $\Omega \subset \mathbb{C}$ given relevant data specified on the boundary $\partial\Omega$. If Ω is geometrically simple then the computations naturally become simpler; in particular the unit disc, the upper half-plane, and annuli have simple geometries with lots of symmetries that facilitate computations. It is therefore desirable to be able to transform problems with complicated underlying geometry into problems involving, e.g., the unit disc.

Conformal mappings are mappings that preserve angles between curves. Loosely speaking this means that they preserve the local geometry up to rotation and scaling. Consequently, problems of geometric nature transform well under conformal mappings, and so it is desirable to be able to map complicated sets to simpler ones by conformal mappings.

7.1 Angles between curves

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two smooth oriented curves intersecting at the point a. We define the *angle* between $\mathcal{C}_1$ and $\mathcal{C}_2$ at a as follows. Let $w_j \neq 0$ be a tangent to $\mathcal{C}_j$ at a pointing in the direction given by the orientation; we identify tangent vectors in $\mathbb{R}^2$ with complex numbers in the usual way. Then the angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ at a is defined by

$$\nu(\mathcal{C}_1, \mathcal{C}_2) = \operatorname{Arg}\frac{w_2}{w_1}.$$

In terms of parametrizations this means that if $\gamma_j \colon I_j \to \mathbb{C}$ is a parametrization of $\mathcal{C}_j$ such that $0 \in I_j$, $\gamma_j(0) = a$, and $|\gamma_j'(0)| = 1$, then $\nu(\mathcal{C}_1, \mathcal{C}_2)$ is the unique number in $(-\pi, \pi]$ such that $e^{i\nu(\mathcal{C}_1,\mathcal{C}_2)}\gamma_1'(0) = \gamma_2'(0)$.

Notice that the angle is a number in the interval $(-\pi, \pi]$. Notice also that if $\nu(\mathcal{C}_1, \mathcal{C}_2) \neq \pi$, then $\nu(\mathcal{C}_1, \mathcal{C}_2) = -\nu(\mathcal{C}_2, \mathcal{C}_1)$; if $\nu(\mathcal{C}_1, \mathcal{C}_2) = \pi$, then $\nu(\mathcal{C}_1, \mathcal{C}_2) = \nu(\mathcal{C}_2, \mathcal{C}_1)$.

Example 7.1. Let $\mathcal{C}_1$ be the unit circle oriented counterclockwise and let $\mathcal{C}_2$ be the imaginary axis oriented upwards. The angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ at $-i$ is $\pi/2$ and the angle at i is $-\pi/2$.

Holomorphic functions with non-zero derivative preserve angles; this is the content of

Theorem 7.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two smooth oriented curves intersecting at a and let f be a holomorphic function in an open set containing a such that $f'(a) \neq 0$. Then $f(\mathcal{C}_1)$ and $f(\mathcal{C}_1)$ are smooth oriented curves in an open set containing $f(a)$ intersecting at $f(a)$, and $\nu(\mathcal{C}_1, \mathcal{C}_2) = \nu(f(\mathcal{C}_1), f(\mathcal{C}_2))$.*

Proof. Let γ_1 and γ_2 be parametrizations as above and set $\Gamma_j(t) = f(\gamma_j(t))$, $j = 1, 2$. Then Γ_j is a smooth parametrization of $f(\mathcal{C}_j)$ and $\Gamma_j(0) = f(\gamma_j(0)) = f(a)$. Moreover, by the Chain rule, $\Gamma_j'(0) = f'(\gamma_j(0)) \cdot \gamma_j'(0) = f'(a) \cdot \gamma_j'(0)$, see Exercise 2.7. Hence,

$$\nu(f(\mathcal{C}_1), f(\mathcal{C}_2)) = \operatorname{Arg}\frac{\Gamma_2'(0)}{\Gamma_1'(0)} = \operatorname{Arg}\frac{f'(a) \cdot \gamma_2'(0)}{f'(a) \cdot \gamma_1'(0)} = \operatorname{Arg}\frac{\gamma_2'(0)}{\gamma_1'(0)} = \nu(\mathcal{C}_1, \mathcal{C}_2).$$

□

Definition 7.3 (Conformal mapping)**.** A *conformal mapping* of an open set $\Omega \subset \mathbb{C}$ is a holomorphic function f defined on Ω such that $f'(z) \neq 0$ for all $z \in \Omega$.

If f is a conformal mapping of Ω then Ω and $f(\Omega)$ have the same local geometry up to rotation and scaling but the global shapes of Ω and $f(\Omega)$ may be quite different. In the next section we will see lots of examples of conformal mappings and how they can change the global shapes.

7.2 Constructions of conformal mappings

We illustrate how one can construct conformal mappings with various properties by composing Möbius transformations, exponential functions, logarithms and powers. Notice that, by the Chain rule, the composition of conformal mappings is again conformal.

Möbius transformations are conformal mappings. Recall that if $f(z) = (az+b)/(cz+d)$ is a Möbius transformation then we require that $ad - bc \neq 0$. Hence,

$$f'(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0,$$

at least for $z \neq -d/c$; if $c = 0$ then f is conformal for all $z \in \mathbb{C}$. With appropriate definitions a Möbius transformation is in fact a conformal mapping of the Riemann sphere, but we will not go further into conformal mappings in this generality.

Example 7.4. Let $f(z) = (z-i)/(z+i)$. Compute the images of the upper half-plane, the lower half-plane, the real axis, the imaginary axis, and the line $\{z;\, \operatorname{Im} z = 1\}$.
Solution: A point z is in the upper half-plane if and only if its distance to i is less than its distance to $-i$, i.e., $|z-i| < |z+i|$. Similarly, z is in the lower half-plane if and only if $|z-i| > |z+i|$ and z is on the real axis if and only if $|z-i| = |z+i|$. Hence, $|f(z)| = |z-i|/|z+i| < 1$ if z is in the upper half-plane, $|f(z)| = 1$ if z is real, and $|f(z)| > 1$ if z is in the lower half-plane. Since Möbius transformations are bijective maps of the Riemann sphere it follows that the image of the upper half-plane is the unit disc $D(0,1)$, the image of the lower half-plane is $\mathbb{C} \setminus D(0,1)$ (together with ∞), and the image of the real axis is the unit circle $\partial D(0,1)$.

It remains to compute the images of the imaginary axis and the line $\{z;\, \operatorname{Im} z = 1\}$. The imaginary axis goes through $-i$, 0, and ∞ so its image must be a circle or line going through $f(-1) = \infty$, $f(0) = -1$, and $f(\infty) = 1$. Hence, the image of the imaginary axis is the real axis. Notice that the image of the imaginary axis intersects the image of the real axis at a right angle. To find the image of the line $\{z;\, \operatorname{Im} z = 1\}$ we notice that it must be a circle since $-i$, the point which is mapped to ∞, is not on the line. Moreover, this circle must contain $f(i) = 0$ and $f(\infty) = 1$ and, in addition, it must intersect the image of the imaginary axis, i.e., the real axis, at a right angle since f is conformal. It follows that the image of $\{z;\, \operatorname{Im} z = 1\}$ is the circle centered at $1/2$ with radius $1/2$. (Draw the picture!)

The exponential function is conformal in $\mathbb{C}$ since $(e^z)' = e^z \neq 0$ by Theorem 6.1. Let S be the vertical strip $\{z;\, a < \operatorname{Re} z < b\}$. The image of S under the exponential function is the annulus $\{w;\, e^a < |w| < e^b\}$ since

$$\exp(S) = \{w = e^z;\, a < \operatorname{Re} z < b\} = \{w = e^x e^{iy};\, a < x < b\} = \{w = re^{iy};\, e^a < r < e^b\}.$$

Similarly we see that the image of a vertical line is a circle centered at 0, and the image of a horizontal line is a ray from 0. Notice in particular that the image of a vertical line intersects the image of a horizontal line at a right angle.

Logarithms are conformal where they are defined since $(\log_\varphi z)' = 1/z$ for $z \in \mathbb{C}_\varphi$; see Chapter 6 for the notation.

Example 7.5. Let $\Omega = \{z;\, \theta_1 < \operatorname{Arg} z < \theta_2\}$, where $-\pi \leq \theta_1 < \theta_2 \leq \pi$. Compute the image of Ω under the principal branch of the logarithm.
Solution: We have

$$\operatorname{Log}(\Omega) = \{w = \operatorname{Log} z;\, \theta_1 < \operatorname{Arg} z < \theta_2\} = \{w = \log|z| + i\operatorname{Arg} z;\, \theta_1 < \operatorname{Arg} z < \theta_2\},$$

and so the image is the horizontal strip $\{w;\, \theta_1 < \operatorname{Im} w < \theta_2\}$.

Power functions $f(z) = z^k$ for $k \in \mathbb{Z}$ are conformal in $\mathbb{C}\setminus\{0\}$ and power functions $f(z) = z^\alpha$ for $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ are conformal where they are defined, see Exercise 7.3. To see what power functions may do to sets consider for instance $f(z) = z^{1/2}$ defined by the principal branch of the logarithm in the cut plane $\mathbb{C}_{-\pi}$. Then

$$\begin{aligned} f(\mathbb{C}_{-\pi}) &= \{w = e^{(\operatorname{Log} z)/2};\ z \in \mathbb{C}_{-\pi}\} = \{w = e^{(\log|z| + i\operatorname{Arg} z)/2};\ -\pi < \operatorname{Arg} z < \pi\} \\ &= \{w = \sqrt{|z|}e^{i(\operatorname{Arg} z)/2};\ -\pi < \operatorname{Arg} z < \pi\} = \{w = \sqrt{|z|}e^{i\theta};\ -\pi/2 < \theta < \pi/2\}, \end{aligned}$$

which is the right half-plane. In a similar way we see that the image of the wedge $\{z = re^{i\theta};\ -\pi/4 < \theta < \pi/4\}$ under the map $f(z) = z^2$ is the right half-plane too. Power functions can thus be used to open up wedges or make them sharper.

We now illustrate how one can construct conformal mappings with certain properties by composing mappings of the above type. Consider a lens-shaped set Ω bounded by two circular arcs C_1 and C_2 starting at 0 and ending at 1; for definiteness say that C_1 is the lower curve and let θ be the angle between C_1 and C_2. Our task is to find a conformal map of Ω onto the unit disc. Viewed on the Riemann sphere there is no essential difference between Ω and the set between two rays going out from the same point with angle θ. More precisely, if we choose a Möbius transformation f_1 that maps $f_1(1) = \infty$ and $f_1(0) = 0$, then $f_1(C_1)$ and $f_1(C_2)$ are rays starting at 0 and $f_1(\Omega)$ is the wedge between $f_1(C_1)$ and $f_1(C_2)$ with angle θ. The choice $f_1(z) = z/(z-1)$ does the job and we set $\Omega_1 := f_1(\Omega)$.

We may open up the the wedge Ω_1 to be a half-plane by using a suitable power. We first make a rotation so that the ray $f_1(C_1)$ becomes the positive real axis; if the angle between the positive real axis and $f_1(C_1)$ is ν then $f_2(z) := e^{-i\nu}z$ accomplishes this. Thus, $\Omega_2 := f_2(\Omega_1)$ is the wedge with angle θ between the positive real axis and the ray $f_2(f_1(C_1))$. To open up Ω_2 to a wedge with angle π we let $f_3(z) := z^{\pi/\theta}$ where we use, e.g., the principal branch of the logarithm to define $z^{\pi/\theta}$. Then one checks (do it!) that $\Omega_3 := f_3(\Omega_2)$ is the upper half-plane.

Finally we let $f_4(z) := (z-i)/(z+i)$, which maps the upper half-plane to the unit disc, see Example 7.4. The composition

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{\big(e^{-i\nu}z/(z-1)\big)^{\pi/\theta} - i}{\big(e^{-i\nu}z/(z-1)\big)^{\pi/\theta} + i}$$

thus is a conformal mapping of Ω onto the unit disc.

Exercises

7.1 Compute the images of the real and imaginary axes and

(a) the lower half-plane under the map $f(z) = (z+i)/(z-i)$,
(b) the right half-plane under the map $f(z) = (z-1)/(z+1)$,
(c) the left half-plane under the map $f(z) = (z+1)/(z-1)$.

7.2 Compute the image under the exponential function of the box

(a) $\{z = x+iy;\ a < x < b, 0 \le y < 2\pi\}$, (b) $\{z = x+iy;\ a < x < b, \theta_1 < y < \theta_2\}$.

7.3 Show that $(z^\alpha)' = \alpha z^{\alpha-1}$ if z^α and $z^{\alpha-1}$ are defined using the same branch of the logarithm.

7.4 Find a conformal mapping of the slit disc $D(0,1) \setminus (-\infty, 0]$ onto $D(0,1)$.

7.5 Find a conformal mapping of the piece of cake $\{z;\ 0 < \operatorname{Arg} z < \pi/2, |z| < 1\}$ onto the upper half-plane.

8 Morera's theorem and Goursat's theorem

We will see that our definition of holomorphicity is equivalent to the "classical" one, i.e., that complex differentiability in an open set implies holomorphicity in our sense. Morera's theorem says that a continuous function whose integral around the boundary of any triangle is 0 must be holomorphic in our sense. Goursat's theorem shows that a function which is complex differentiable in an open set satisfies the hypothesis of Morera's theorem.

8.1 Morera's theorem

By a triangle in an open set Ω we mean a bounded open subset $\Delta \subset \Omega$ such that $\overline{\Delta} \subset \Omega$ and the boundary $\partial\Delta$ of Δ consists of three line segments.

Theorem 8.1 (Morera's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let f be a continuous function in Ω such that $\int_{\partial\Delta} f(z)\,dz = 0$ for all triangles Δ in Ω. Then f is holomorphic in Ω.*

Proof. Holomorphicity is a local property so it suffices to show that, given any point $z_0 \in \Omega$, f is holomorphic in some disc $D(z_0, r)$, $r > 0$. Let $z_0 \in \Omega$ and choose $r > 0$ such that $D(z_0, r) \subset \Omega$; this is possible since Ω is open. We define a function F in $D(z_0, r)$ by setting

$$F(z) = \int_{[z_0,z]} f(z)\,dz,$$

where $[z_0, z]$ is the line segment starting at z_0 and ending at z. Fix $w \in D(z_0, r)$ and let h be a complex number with $|h|$ so small that the triangle with corners at z_0, w, and $w + h$ is contained in $D(z_0, r)$. Since the integral of f around the boundary of any triangle in Ω is 0 we have

$$0 = \int_{[z_0,w]} f(z)\,dz + \int_{[w,w+h]} f(z)\,dz + \int_{[w+h,z_0]} f(z)\,dz.$$

Hence we get

$$\begin{aligned} F(w+h) - F(w) &= \int_{[z_0,w+h]} f(z)\,dz - \int_{[z_0,w]} f(z)\,dz = \int_{[w,w+h]} f(z)\,dz \\ &= \int_{t=0}^{1} f(w+th)\,hdt = h\int_0^1 \big(f(w) + f(w+th) - f(w)\big)\,dt \\ &= h \cdot f(w) + h\int_0^1 \big(f(w+th) - f(w)\big)\,dt, \end{aligned}$$

where we in the third step have parametrized $[w, w + h]$ by $[0, 1]$ using $t \mapsto w + th$. Since f is continuous, $|f(w + th) - f(w)| \to 0$ as $t \to 0$ and so we see that

$$F(w + h) - F(w) = h \cdot f(w) + o(|h|).$$

It follows that F is complex differentiable in $D(z_0, r)$ with $F'(w) = f(w)$. Moreover, $F \in C^1(D(z_0, r))$ since it also follows that $-i\partial F/\partial y = \partial F/\partial x = f(w)$, which is continuous (cf. the Cauchy-Riemann equations). Thus, F is holomorphic in $D(z_0, r)$ and so, by Corollary 4.10, $f = F'$ is holomorphic in $D(z_0, r)$. $\square$

8.2 Goursat's theorem

Theorem 8.2 (Goursat's theorem)**.** *Let f be a function which is complex differentiable in an open set $\Omega \subset \mathbb{C}$. Then $\displaystyle\int_{\partial\Delta} f(z)\,dz = 0$ for any triangle $\Delta \subset \Omega$.*

Proof. We notice first that f is continuous in Ω, see Exercise 2.4. Let $\Delta_0 \subset \Omega$ be a triangle and set $I := \int_{\partial\Delta_0} f(z)\,dz$; we will show that $I = 0$. Divide Δ_0 into four triangles Δ_0^j, $j = 1, \dots, 4$, by inscribing a triangle in Δ_0 with corners on the midpoints of the edges of Δ_0. Then, Exercise 8.1,

$$\int_{\partial\Delta_0} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial\Delta_0^j} f(z)\,dz, \tag{8.1}$$

and so there must be at least one j such that $|\int_{\partial\Delta_0^j} f(z)\,dz| \geq |I|/4$. Let Δ_1 be one of the Δ_0^j's with this property. Notice that the circumference of Δ_1, $\ell(\partial\Delta_1)$, is half that of Δ_0. Repeating this process of dividing a triangle into four smaller triangles we get a decreasing sequence of triangles, $\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \cdots$, such that

$$\left|\int_{\partial\Delta_k} f(z)\,dz\right| \geq |I|/4^k \quad \text{and} \quad \ell(\partial\Delta_k) = \ell(\partial\Delta_0)/2^k.$$

Let $a \in \cap_k \overline{\Delta}_k$; it is a fact of topology that a decreasing sequence of compact subsets has non-empty intersection. Since f is complex differentiable at a there is an A (the limit of $(f(a+h) - f(a))/h$ as $h \to 0$) such that $f(a+h) - f(a) = Ah + o(|h|)$. Hence, for any given $\epsilon > 0$ there is a $\delta > 0$ such that if $|z - a| < \delta$ then

$$\big|f(z) - f(a) - A(z-a)\big| \leq \epsilon|z-a|.$$

Take k so large that $\overline{\Delta}_k \subset D(a, \delta)$; this is possible since the Δ_j's shrink to a. By Cauchy's theorem $\int_{\partial\Delta_k} (f(a) + A(z-a))\,dz = 0$ and so we get

$$\begin{aligned}
\left|\int_{\partial\Delta_k} f(z)\,dz\right| &= \left|\int_{\partial\Delta_k} \big(f(z) - f(a) - A(z-a)\big)\,dz\right| \\
&\leq \sup_{z\in\partial\Delta_k} |f(z) - f(a) - A(z-a)| \cdot \ell(\partial\Delta_k) \\
&\leq \sup_{z\in\partial\Delta_k} \epsilon|z-a|\ell(\partial\Delta_k) \\
&\leq \epsilon \cdot \ell(\partial\Delta_k)^2 = \epsilon \cdot \ell(\partial\Delta_0)^2/4^k.
\end{aligned}$$

Thus, for k large enough, $|I|/4^k \leq \displaystyle\int_{\partial\Delta_k} f(z)\,dz \leq \epsilon \cdot \ell(\partial\Delta_0)^2/4^k$ so $|I| \leq \epsilon \cdot \ell(\partial\Delta_0)^2$. But $\epsilon > 0$ is arbitrary so $|I| = 0$, i.e., $I = 0$. □

Exercises

8.1 Show that (8.1) holds.

8.2 Let $\Omega \subset \mathbb{C}$ be open and let $\{f_n\}_n$ be a sequence of holomorphic functions in Ω. Show that if $f_n(z)$ converges uniformly on each compact subset of Ω to some function $f(z)$, then f is holomorphic in Ω. (Hint: Use Exercise 2.8 to show that f satisfies the hypothesis of Morera's theorem.)

9 Zeros of holomorphic functions

A polynomial of degree at most n cannot have more than n zeros unless it is identically zero. As we will see, something similar holds for holomorphic functions; a holomorphic function cannot have "too many" zeros unless it is identically zero. The reason is, loosely speaking, that a holomorphic function locally is given by a convergent power series, which can be thought of as a polynomial of infinite degree. This means that holomorphic functions, just as polynomials, are rigid objects; two holomorphic functions which agree at sufficiently many points have to agree everywhere. This is in stark contrast to smooth or continuous functions which are much more plastic.

9.1 Zeros, their orders, and rigidity theorems

Let $\Omega \subset \mathbb{C}$ and let f be a holomorphic function in Ω. We let $Z(f) := \{a \in \Omega;\ f(a) = 0\}$ be the zero set of f. We say that $a \in Z(f)$ has *multiplicity* m, or that a is a *zero of order* m, if

$$f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0, \quad f^{(m)}(a) \neq 0.$$

Example 9.1. Let $f(z) = z^2 - 1$. Then $Z(f) = \{1, -1\}$ and both 1 and -1 are zeros of order 1 since $f(\pm 1) = 0$ and $f'(\pm 1) = \pm 2 \neq 0$.

Example 9.2. The function $\cos z - 1$ has a zero of order 2 at $z = 0$ since $\cos 0 - 1 = 0$, $(\cos z - 1)'(0) = -\sin 0 = 0$, and $(\cos z - 1)''(0) = -\cos 0 = -1 \neq 0$.

A holomorphic function cannot have a zero of infinite order unless the function is identically zero. More precisely, we have

Theorem 9.3 (Identity theorem I)**.** *Let $\Omega \subset \mathbb{C}$ be a connected open set and let f be holomorphic in Ω. If there is an $a \in \Omega$ such that $f^{(k)}(a) = 0$ for all integers $k \geq 0$, then f is identically 0 in Ω.*

Proof. Consider the set S of points $a \in \Omega$ such that $f^{(k)}(a) = 0$ for all integers $k \geq 0$. Then S is a closed subset of Ω since all $f^{(k)}$ are continuous. On the other hand, by Theorem 4.7,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k \tag{9.1}$$

in some disc $D(a, r)$ and so, if $a \in S$, then $f = 0$ in some open disc centered at a. This disc then is a subset of S and so S open. Hence, both S and $\Omega \setminus S$ are open subsets of Ω, and since Ω is connected either S or $\Omega \setminus S$ has to be empty, see Appendix B. Thus, if there is an $a \in S$, then $S = \Omega$ and so f is 0 in Ω. □

Example 9.4. Let Ω be an open connected set, $\mathcal{C}$ a curve in Ω, and f a holomorphic function in Ω. Show that if $f = 0$ on $\mathcal{C}$, then $f = 0$ in Ω.
Solution: Let $z_0 \in \mathcal{C}$. Since $f'(z_0)$ is given by the limit (2.5) it follows by letting $h \to 0$ along $\mathcal{C}$ that $f'(z_0) = 0$. But $z_0 \in \mathcal{C}$ is arbitrary so $f' = 0$ on $\mathcal{C}$. Repeating the argument it follows that $f^{(k)} = 0$ on $\mathcal{C}$ for all $k \geq 0$. By the Identity theorem I, $f = 0$ in Ω.

Proposition 9.5. *Let f be holomorphic in a disc $D(a, r)$. Then the following are equivalent.*

(a) *f has a zero of order m at a.*

(b) *There is a holomorphic function g in $D(a, r)$ such that $g(a) \neq 0$ and $f(z) = (z-a)^m g(z)$.*

(c) *The limit* $\lim_{z\to a} f(z)/(z-a)^m$ *exists and is non-zero.*

Proof. Let ℓ be the least integer such that $f^{(\ell)}(a) \neq 0$. Then f has a zero of order m at a if and only if $\ell = m$. Moreover, by Theorem 4.7, see also (9.1), we have

$$f(z) = (z-a)^\ell \cdot \left(\frac{f^{(\ell)}(a)}{\ell!} + \frac{f^{(\ell+1)}(a)}{(\ell+1)!}(z-a) + \frac{f^{(\ell+2)}(a)}{(\ell+2)!}(z-a)^2 + \cdots \right). \tag{9.2}$$

Thus, if f has a zero of order m at a then $f(z) = (z-a)^m g(z)$ and $g(a) = f^{(m)}(a)/m! \neq 0$. Hence, (a) $\Rightarrow$ (b).

To see that (b) $\Rightarrow$ (c) we just notice that if $f(z) = (z-a)^m g(z)$ with $g(a) \neq 0$, then $\lim_{z\to a} f(z)/(z-a)^m = g(a) \neq 0$.

It remains to see that (c) $\Rightarrow$ (a). Assume that $\lim_{z\to a} f(z)/(z-a)^m$ exists and is non-zero, and let ℓ be the number defined above; we'll show that $\ell = m$. In view of (9.2), if $\ell < m$ then $|f(z)|/|z-a|^m \to \infty$ as $z \to a$ and so $\lim_{z\to a} f(z)/(z-a)^m$ cannot exist; if $\ell > m$ then $\lim_{z\to a} f(z)/(z-a)^m = 0$. Hence, $\ell = m$ and we are done. □

Example 9.6. Compute the multiplicity of the zero at $z = 0$ of $\sin^{12} z$.
Solution: $\sin z$ has a zero of order 1 at $z = 0$ since $\sin 0 = 0$ and $(\sin z)'(0) = \cos 0 = 1 \neq 0$. By Proposition 9.5, $\sin z = zg(z)$ where g is holomorphic and $g(0) \neq 0$. Hence, $\sin^{12} z = z^{12} g(z)^{12}$. Since $g(0)^{12} \neq 0$, Proposition 9.5 shows that $\sin^{12} z$ has a zero of multiplicity 12 at $z = 0$.

The next result, which can be seen as a jazzed-up version of the Identity theorem I, says that a non-trivial holomorphic function must have isolated zeros. That a function in some open set has an isolated zero at a means that for some $r > 0$ the only zero of f in the disc $D(a,r)$ is a.

Theorem 9.7 (Identity theorem II)**.** *Let f be a holomorphic function in an open connected set $\Omega \subset \mathbb{C}$. Then either f is identically 0 in Ω or f has only isolated zeros in Ω.*

Proof. Let E be the set of limit points of $Z(f)$ in Ω (see Appendix B); it is an exercise (do it!) to show that $E \subset Z(f)$. The set E thus is the set of non-isolated zeros of f in Ω. We will show that either f is identically 0 or that E is empty, in which case f only has isolated zeros.

Assume that E is non-empty and let $a \in E$. If f is not identically zero, then a must be a zero of f with some finite multiplicity m by the Identity theorem I. Thus, by Proposition 9.5, $f(z) = (z-a)^m g(z)$ in some open disc centered at a where g is holomorphic and $g(a) \neq 0$. Since g in particular is continuous, $g \neq 0$ in some open set containing a and so there is a disc $D(a,r)$, $r > 0$, such that $g \neq 0$ in $D(a,r)$. But then a is an isolated zero of f. This contradicts that $a \in E$ and so E must be empty. □

Corollary 9.8. *Let $\Omega \subset \mathbb{C}$ be open and connected and let f and g be holomorphic functions in Ω. If there is a set $E \subset \Omega$ that has a limit point in Ω and $f(z) = g(z)$ for $z \in E$, then $f = g$ in Ω.*

Proof. The function $F(z) := f(z) - g(z)$ is holomorphic in Ω and $Z(F)$ contains E. Thus, F has non-isolated zeros in Ω and must therefore be identically zero in Ω by the Identity theorem II. □

9.2 Analytic continuation

Let f be a holomorphic function in an open set $\Omega \subset \mathbb{C}$. Suppose that $\tilde{f}$ is a holomorphic function in some open set $\tilde{\Omega}$ and that $f = \tilde{f}$ in $\Omega \cap \tilde{\Omega}$. Then we can extend f to a holomorphic function in $\Omega \cup \tilde{\Omega}$ by setting $f(z) := \tilde{f}(z)$ for $z \in \tilde{\Omega}$. So far this has nothing to do with f and $\tilde{f}$ being holomorphic but the point is that if $\Omega \cup \tilde{\Omega}$ is connected, then f can be extended to a holomorphic function in $\Omega \cup \tilde{\Omega}$ in only one way. In fact, since Ω is open any point of Ω is a limit point of Ω and so, if $\Omega \cup \tilde{\Omega}$ is connected, any holomorphic extension of f to $\Omega \cup \tilde{\Omega}$ must agree with $\tilde{f}$ in $\tilde{\Omega}$ by Corollary 9.8.

If a holomorphic function f a priori defined in some open set Ω_0 can be extended to a holomorphic function in some bigger open set Ω_1 we say that f can be *analytically continued* to Ω_1; if Ω_1 is connected the analytic continuation of f is unique.

Example 9.9. The power series $\sum_{k=0}^{\infty} z^k$ has radius of convergence 1 and thus defines a holomorphic function f in $D(0,1)$. But $\sum_{k=0}^{\infty} z^k = 1/(1-z)$ in $D(0,1)$ and $1/(1-z)$ is holomorphic in $\mathbb{C} \setminus \{1\}$ so f can be analytically continued to $\mathbb{C} \setminus \{1\}$.

Holomorphic functions cannot always be analytically continued but there is a, at least theoretically, natural way to find the analytic continuation if it exists. Suppose that f is holomorphic in some open set Ω and let $a \in \Omega$. Let r be the distance from a to the boundary $\partial\Omega$. Then, by Theorem 4.7, f is represented by its Taylor series in $D(a,r)$. Now if this series happens to converge in a bigger disc then f can be analytically continued beyond Ω. On the other hand, if f can be analytically continued across the boundary points of Ω closest to a, then the Taylor series will converge in a larger disc by Theorem 4.7.

Possibly unexpected things might happen in connection with analytic continuation. As an illustration, consider the restriction of $\operatorname{Log} z$ to the set $\{z; -\pi/2 < \operatorname{Arg} z < \pi\}$. Obviously it can be extended across the negative imaginary axis to the third quarter as the principal branch of the logarithm. But it can also be extended across the negative real axis to the third quarter by defining it to be $\log_{-\pi/2} z$ for z in the left half-plane. These two extensions do not agree in the third quarter but differ by $2\pi i$. How come this doesn't contradict Corollary 9.8 or the uniqueness of analytic continuation?

9.3 Counting the number of zeros

Let $\Omega \subset \mathbb{C}$ be a bounded open set and assume that the boundary $\partial\Omega$ is a finite union of piecewise smooth simple closed curves. Then

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{dz}{z} = \begin{cases} 1 & \text{if } 0 \in \Omega \\ 0 & \text{if } 0 \notin \overline{\Omega} \end{cases} . \tag{9.3}$$

In fact, if $0 \notin \overline{\Omega}$ then $1/z$ is holomorphic in some open set containing $\overline{\Omega}$ and so the integral is 0 by Cauchy's theorem. On the other hand, if $0 \in \Omega$ we choose a small disc $\overline{D(0,r)} \subset \Omega$ and apply Cauchy's theorem to $\Omega \setminus \overline{D(0,r)}$ to see that the integral is equal to $\int_{\partial D(0,r)} dz/z$, which equals $2\pi i$, see Example 2.4.

One way to think of (9.3) is that the left-hand side counts the number of zeros of the function $f(z) = z$ in Ω. Another interpretation is that the left-hand side computes the total number of times the boundary $\partial\Omega$ winds around the origin; this is the same as the total change in argument, divided by 2π, that a point traveling along $\partial\Omega$ experiences.

The following result is a generalization of (9.3).

Theorem 9.10 (Argument principle)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded open set whose boundary $\partial\Omega$ consists of a finite number of piecewise smooth simple closed curves. If f is holomorphic in an open set containing $\overline{\Omega}$ and $f(z) \neq 0$ for $z \in \partial\Omega$, then*

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)\,dz}{f(z)} \tag{9.4}$$

equals the number of zeros of f in Ω counted with multiplicity.

Proof. First notice that the set of zeros of f in Ω must be finite because if not, then, since Ω is bounded, the zero set of f in Ω would have a limit point in $\overline{\Omega}$ by the Bolzano-Weierstrass theorem (google it!) and so f would be identically zero in at least one connected component of Ω. But this is impossible since $f(z) \neq 0$ for $z \in \partial\Omega$. Let $a_1, \dots, a_n$ be the distinct zeros of f in Ω and let their multiplicities by $m_1, \dots, m_n$, respectively.

If $r > 0$ is sufficiently small then the discs $\overline{D(a_1, r)}, \dots, \overline{D(a_n, r)}$ are pairwise disjoint and contained in Ω. Then $f'(z)/f(z)$ is holomorphic in an open set containing $\overline{\Omega} \setminus \cup_k D(a_k, r)$ and so Cauchy's theorem shows that

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)\,dz}{f(z)} = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\partial D(a_k,r)}\frac{f'(z)\,dz}{f(z)}.$$

To compute the right-hand side we note that by Proposition 9.5, $f(z) = (z - a_k)^{m_k} \cdot g_k(z)$ in $D(a_k, r)$ where g_k is holomorphic and $g_k(a_k) \neq 0$. Shrinking r if necessary we may assume that $g_k \neq 0$ in $\overline{D(a_k, r)}$. Thus

$$\frac{f'(z)}{f(z)} = \frac{m_k(z - a_k)^{m_k-1}g_k(z) + (z - a_k)^{m_k}g_k'(z)}{(z - a_k)^{m_k}g_k(z)} = m_k\frac{1}{z - a_k} + \frac{g_k'(z)}{g_k(z)}$$

in $D(a_k, r) \setminus \{a_k\}$ and so we get

$$\frac{1}{2\pi i}\int_{\partial D(a_k,r)}\frac{f'(z)\,dz}{f(z)} = \frac{m_k}{2\pi i}\int_{\partial D(a_k,r)}\frac{dz}{z - a_k} + \frac{1}{2\pi i}\int_{\partial D(a_k,r)}\frac{g_k'(z)\,dz}{g_k(z)} = m_k,$$

where the last equality follows from Cauchy's formula (applied to the function 1) and Cauchy's theorem (applied to g'/g) noticing that g'/g is holomorphic in an open set containing $\overline{D(a_k, r)}$. Hence, (9.4) equals $m_1 + \cdots + m_n$ and the theorem follows. □

We'll say some words hopefully explaining the name "Argument principle". Assume for simplicity that $\partial\Omega$ consists of one simple smooth closed curve and let $\gamma\colon [0, 1) \to \mathbb{C}$ be a simple smooth parametrization of it. Then $f \circ \gamma$ is a parametrization of the curve $f(\partial\Omega)$. But $f(\partial\Omega)$ need not be simple and, moreover, $f \circ \gamma(t)$ might run through $f(\partial\Omega)$ several times as t runs through $[0, 1)$. We claim that (9.4) equals the total number of times that $f \circ \gamma(t)$ winds around the origin as t runs through $[0, 1)$; this is the same as the total change in argument, divided by 2π, that $f \circ \gamma(t)$ experiences as t runs through $[0, 1)$, cf. the paragraph preceding Theorem 9.10. To see this we first notice that this "winding number" is computed by

$$\frac{1}{2\pi i}\int_{t=0}^{1}\frac{(f \circ \gamma)'(t)\,dt}{f \circ \gamma(t)},$$

cf. the beginning of this section. By the Chain rule, $(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$ (see Exercise 2.7), and so

$$\frac{1}{2\pi i}\int_{t=0}^{1}\frac{(f \circ \gamma)'(t)\,dt}{f \circ \gamma(t)} = \frac{1}{2\pi i}\int_{t=0}^{1}\frac{f'(\gamma(t)) \cdot \gamma'(t)\,dt}{f(\gamma(t))} = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)\,dz}{f(z)}$$

since γ is a simple parametrization of $\partial\Omega$.

The next result, Rouche's theorem, which is a consequence of the Argument principle, is often useful to compute/estimate the number of zeros in concrete examples.

Theorem 9.11 (Rouche's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded open set whose boundary $\partial\Omega$ consists of a finite number of piecewise smooth simple closed curves. Let f and g be holomorphic in an open set containing $\overline{\Omega}$ and assume that $|f(z)| > |g(z)|$ for $z \in \partial\Omega$. Then f and $f+g$ have the same number of zeros in Ω, multiplicities taken into account.*

Proof. Consider the function

$$\varphi(t) := \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)+tg'(z)}{f(z)+tg(z)}\,dz$$

defined for $t \in [0,1]$. Notice that $|f(z)+tg(z)| \geq |f(z)| - t|g(z)| \geq |f(z)| - |g(z)| > 0$ for all $(z,t) \in \partial\Omega \times [0,1]$. It follows that there is some $c > 0$ such that $|f(z)+tg(z)| \geq c$ for all $(z,t) \in \partial\Omega\times[0,1]$ (why?). By the Argument principle, $\varphi(t)$ is the number of zeros of $f(z)+tg(z)$ in Ω, counting multiplicities. In particular, $\varphi(0)$ is the number of zeros of f in Ω and $\varphi(1)$ is the number of zeros of $f+g$ in Ω.

We claim that φ is continuous. Given the claim we have that φ is a continuous function on $[0,1]$ taking integer values. But then φ must be constant by the Intermediate value theorem and so $\varphi(0) = \varphi(1)$. It remains to show the claim. To do this we make a straightforward computation to get

$$\begin{aligned}
\varphi(t) - \varphi(t_0) &= \frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{f'(z)+tg'(z)}{f(z)+tg(z)} - \frac{f'(z)+t_0g'(z)}{f(z)+t_0g(z)}\right)dz \\
&= \frac{t-t_0}{2\pi i}\int_{\partial\Omega}\frac{g'(z)f(z)-f'(z)g(z)}{\big(f(z)+tg(z)\big)\big(f(z)+t_0g(z)\big)}\,dz.
\end{aligned}$$

Since $g'(z)f(z) - f'(z)g(z)$ is continuous there is a $C > 0$ such that $|g'(z)f(z)-f'(z)g(z)| \leq C$ for all $z \in \partial\Omega$ and so the absolute value of the integrand is $\leq C/c^2$ for all $(z,t) \in \partial\Omega \times [0,1]$. By Proposition 2.7 we thus see that $\varphi(t) - \varphi(t_0) = O(|t-t_0|)$ and the claim follows. $\square$

Example 9.12. Find the number of zeros of $z^5 - 3z + 1$ in the unit disc.
Solution: Set $f(z) = -3z+1$ and $g(z) = z^5$. Clearly, the only zero of f is $z = 1/3$ which is in $D(0,1)$. For z on the boundary of $D(0,1)$, i.e., $|z| = 1$, we have

$$|f(z)| = |-3z+1| \geq |-3z| - |1| = 2 > 1 = |z|^5 = |g(z)|.$$

By Rouche's theorem, $f(z)$ and $f(z)+g(z) = z^5 - 3z + 1$ have the same number of zeros in $D(0,1)$ and so $z^5 - 3z + 1$ has one zero in $D(0,1)$.

Exercises

9.1 Show, using Identity theorem I or II, that

$$\text{(a)}\ \ \sin^2 z + \cos^2 z = 1, \quad \text{(b)}\ \ \sin(2z) = 2\sin z\cos z, \quad \text{(c)}\ \ \cos(2z) = \cos^2 z - \sin^2 z,$$

for all $z \in \mathbb{C}$.

9.2 Find the zeros and their multiplicities of (a) $(z^2-1)\sin\pi z$ and (b) $e^{z^2} - 1$.

9.3 Let $\{z_n\} \subset D(0,1)$ be a sequence of points $\neq 0$ such that $z_n \to 0$ as $n \to \infty$. Is there a holomorphic function f in $D(0,1)$ such that $f(z_n) = (-1)^n z_n$ for all n?

9.4 Is there a holomorphic function in Ω such that $f(0) = 1$ and $f(1 - 1/2n) = 0$ for $n = 1, 2, \dots$ if (a) $\Omega = \mathbb{C}$, (b) Ω is the unit disc?

9.5 Find all holomorphic functions f in $\mathbb{C}$ satisfying $f(1/n) = n^2 f(1/n)^3$ for $n = 1, 2, \dots$.

9.6 Find the number of zeros of

(a) $5z^6 + z^4 - z + 2$ in the unit disc,

(b) $z^4 + z^2 + 3z + 5.5$ in the annulus $\{z;\ 1 < |z| < 2\}$,

(c) $z^4 + z^2 + 5z$ in the annulus $\{z;\ 1 < |z| < 2\}$,

(d) $\cos(\pi z) - 100z^{100}$ in the unit disc.

9.7 Let R be a given positive number. Show that $\displaystyle\sum_{k=0}^{N} \frac{z^k}{k!} \neq 0$ for all $z \in D(0, R)$ if N is sufficiently large.

10 Some topology and jazzed-up Cauchy theorems

We'll get a glimpse of the connection between topological properties of an open set and analytic properties of holomorphic functions in that set.

10.1 Deformation and homotopy

Let $\Omega \subset \mathbb{C}$ be open and let γ_0 and γ_1 be parametrized closed curves in Ω. A closed curve can be parametrized by the interval $[0,1]$ and in this chapter we will tacitly assume that all parametrized closed curves are parametrized by $[0,1]$. The curves γ_0 and γ_1 are said to be *homotopic* in Ω if there is a continuous map $H\colon [0,1]\times[0,1] \to \Omega$ such that

$$H(s,0) = \gamma_0(s), \qquad H(s,1) = \gamma_1(s), \qquad H(0,t) = H(1,t) \ \ \forall t \in [0,1].$$

The last condition ensures that $s \mapsto H(s,t)$ is a closed curve. One should think of the curves $s \mapsto H(s,t)$ as interpolating continuously between γ_0 and γ_1 in Ω as t goes from 0 to 1, i.e., that γ_0 is continuously deformed inside Ω to γ_1 as t goes from 0 to 1. We will call the map H a homotopy between γ_0 and γ_1.

The notion of two curves being homotopic turns out to be an equivalence relation on the set of parametrized closed curves in Ω and thus gives a partition of this set into equivalence classes, called homotopy classes. A connected open set such that all curves are homotopic, i.e., the number of homotopy classes is 1, is called *simply connected*. Intuitively one can think of a simply connect open set as having "no holes".

If a is a point in an open set Ω then the constant map $\gamma(s) = a$ for all $s \in [0,1]$ is a parametrized closed curve in Ω. A parametrized closed curve in Ω is said to be *null-homotopic* in Ω if it is homotopic to a point in Ω, viewed as such a trivial curve. Notice that an open connected set is simply connected if and only if every parametrized closed curve is null-homotopic.

Example 10.1. Let Ω be an open convex set. Then Ω is simply connected. In fact, let γ be a closed parametrized curve in Ω and let $a \in \Omega$. Since Ω is convex, the line segment between a and $\gamma(s)$ is contained in Ω for all $s \in [0,1]$, and so $H(s,t) := \gamma(s)\cdot(1-t) + ta \in \Omega$ for all $(s,t) \in [0,1]\times[0,1]$. Since H clearly depends continuously on (s,t), $H(s,0) = \gamma(s)$, $H(s,1) = a$, and $H(0,t) = \gamma(0)(1-t) + ta = \gamma(1)(1-t) + ta = H(1,t)$ it follows that $H\colon [0,1]\times[0,1] \to \Omega$ is a homotopy between γ and the trivial curve a.

If f is holomorphic in Ω and γ is a parametrized smooth closed curve in Ω we set

$$\int_\gamma f(z)\,dz := \int_{t=0}^{1} f\big(\gamma(t)\big)\gamma'(t)dt;$$

if γ is merely piecewise smooth we divide γ into smooth pieces and add up the integrals. We call such integrals Cauchy integrals; the content of the following result is that Cauchy integrals only depend on the homotopy class of the curve.

Theorem 10.2 (Cauchy's theorem, homotopy version)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let γ_0 och γ_1 be homotopic parametrized closed piecewise smooth curves in Ω. Then, for any holomorphic function f in Ω,*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

In particular, if Ω is simply connected, then $\int_\gamma f(z)\,dz = 0$ for any holomorphic function f in Ω and any parametrized closed piecewise smooth curve γ in Ω.

Sketch of proof. The last statement follows from the first since $\int_\gamma f(z)\,dz = 0$ for any function if γ is constant.

To prove the first we take a homotopy H between γ_0 and γ_1. We divide $[0,1]$ into n pieces of equal length. This gives us a grid of $(n+1)\cdot(n+1)$ points $(i/n, j/n)$, $i,j = 0,\dots,n$, in $[0,1]\times[0,1]$. Consider the images $H\big(i/n, j/n\big)$ in Ω of these points. Let ℓ_{ij}, for $i = 0,1,\dots,n$ and $j = 0,1,\dots,n-1$, be the oriented line segment starting at $H\big(i/n, j/n\big)$ and ending at $H\big(i/n,(j+1)/n\big)$; notice that $\ell_{0j} = \ell_{nj}$. Let Γ_{ij}, for $i = 0,1,\dots,n-1$ and $j = 1,\dots,n-1$, be the oriented line segment starting at $H\big(i/n, j/n\big)$ and ending at $H\big((i+1)/n, j/n\big)$; we let Γ_{i0} be the curve-piece $\gamma_0([i/n,(i+1)/n])$ and Γ_{in} be the curve-piece $\gamma_1([i/n,(i+1)/n])$. Since H is continuous and Ω is open ℓ_{ij} and Γ_{ij} are contained in Ω if n is large enough. In fact, since $H([0,1]\times[0,1])$ is a compact subset of Ω (why?) it has a positive distance to $\partial\Omega$ and so we may choose n so large that all ℓ_{ij} and Γ_{ij} are contained in Ω for all i and j.

We get oriented closed curves P_{ij}, for $i = 0,\dots,n-1$ and $j = 0,\dots,n-1$ by joining Γ_{ij}, $\ell_{i+1,j}$, $-\Gamma_{i,j+1}$, and $-\ell_{ij}$ in this order. It is an exercise (do it!) to use Cauchy's theorem and check that

$$\int_{P_{ij}} f(z)\,dz = 0,\ \forall i,j = 0,\dots,n-1.$$

We also get oriented closed curves Γ_j, for $j = 0,1,\dots,n$, by joining $\Gamma_{0j},\Gamma_{1j},\dots,\Gamma_{n-1,j}$ in this order. Notice that Γ_0 and Γ_n are (the images of) γ_0 and γ_1 respectively. The theorem will follow if we can show that

$$\int_{\Gamma_{j+1}} f(z)\,dz - \int_{\Gamma_j} f(z)\,dz = 0, \tag{10.1}$$

for $j = 0,\dots,n-1$. To show (10.1) we add and subtract integrals along the lines ℓ_{ij}:

$$\begin{aligned}
\int_{\Gamma_{j+1}} f(z)\,dz - \int_{\Gamma_j} f(z)\,dz &= \sum_{i=0}^{n-1}\int_{\Gamma_{i,j+1}} f(z)\,dz - \int_{\Gamma_{ij}} f(z)\,dz + \int_{\ell_{ij}} f(z)\,dz - \int_{\ell_{i+1,j}} f(z)\,dz \\
&\quad -\sum_{i=0}^{n-1}\int_{\ell_{ij}} f(z)\,dz - \int_{\ell_{i+1,j}} f(z)\,dz \\
&= -\int_{P_{ij}} f(z)\,dz - 0 = 0,
\end{aligned}$$

where the second last equality follows since $\ell_{0j} = \ell_{nj}$, which makes the sum $\sum_{i=0}^{n-1}\int_{\ell_{ij}} f(z)\,dz - \int_{\ell_{i+1,j}} f(z)\,dz$ telescoping. □

10.2 Winding numbers and homology

In the previous section we saw that Cauchy integrals only depend on the homotopy class of the curve. In particular, $\displaystyle\int_\gamma f(z)dz = 0$ if γ is null-homotopic in the set where f is holomorphic. But homotopy is not the whole story. If Ω is $\mathbb{C}$ with two distinct points removed then one can find a curve γ in Ω (google "null homologous" and "picture") that is not null-homotopic in Ω, but still $\displaystyle\int_\gamma f(z)dz = 0$ for any holomorphic function f i Ω. The "reason" for this is that the total number of times that γ winds around any of the points outside of Ω is 0. This is loosely speaking the content of the homology version of Cauchy's theorem.

Let $\Omega \subset \mathbb{C}$ be an open set. A *cycle* in Ω is a formal finite sum $\Gamma = \sum_k n_k\mathcal{C}_k$ where $n_k \in \mathbb{Z}$ and $\mathcal{C}_k$ is a piecewise smooth oriented closed curve in Ω. The *support*, $|\Gamma|$, of Γ is the union of the $\mathcal{C}_k$'s. If f is a continuous function on $|\Gamma|$ then we set

$$\int_\Gamma f(z)\,dz := \sum_k n_k \int_{\mathcal{C}_k} f(z)\,dz.$$

We let $\ell(\Gamma) := \sum_k |n_k|\ell(\mathcal{C}_k)$ be the length of Γ. One should think of the cycle $\Gamma = \sum_k n_k\mathcal{C}_k$ as the curves $\mathcal{C}_k$ run through n_k times; if n_k is negative this means that $\mathcal{C}_k$ should be run through $|n_k|$ times in the direction opposite to the one given by the orientation.

Example 10.3. Let Ω be an open set with piecewise smooth boundary $\partial\Omega$. Then $\partial\Omega$, oriented as a boundary, (i.e., Ω is on the left-hand side) is a cycle in $\mathbb{C}$. If f is holomorphic in an open set containing $\overline{\Omega}$, then $f(\partial\Omega)$ has a natural structure as a cycle. For instance, if $\Omega = D(0,1)$ and $f(z) = z^n$ then $f(\partial D(0,1)) = n\partial D(0,1)$.

Definition 10.4. If Γ is a cycle in $\mathbb{C}$ and $a \in \mathbb{C}\setminus|\Gamma|$ we define the *winding number*, or *index*, denoted $\mathrm{Ind}_\Gamma(a)$, of Γ with respect to a as

$$\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}.$$

The interpretation of $\mathrm{Ind}_\Gamma(a)$ is as the total number of times that Γ winds around a. Some evidence for this interpretation is given in Section 9.3. Further evidence is given by the following result.

Theorem 10.5. *Let Γ be a cycle in $\mathbb{C}$.*

(a) *For each $a \in \mathbb{C}\setminus|\Gamma|$, $\mathrm{Ind}_\Gamma(a)$ is an integer.*

(b) *The function $a \mapsto \mathrm{Ind}_\Gamma(a)$, defined in $\mathbb{C}\setminus|\Gamma|$, is constant on each connected component of $\mathbb{C}\setminus|\Gamma|$. Moreover, $\mathrm{Ind}_\Gamma(a) = 0$ if $|a|$ is sufficiently large.*

Proof. To prove (a) we may assume that Γ is a piecewise smooth oriented closed curve (why?) and that $a = 0$; we should then show that $\int_\Gamma dz/z$ is $2\pi i$ times an integer. Choose points $w_1, w_2, \dots, w_N$ on Γ and discs $D(w_j, r)$, $j = 1, \dots, N-1$, such that (i) $w_1 = w_N$, (ii) $\Gamma \subset \cup_j D(w_j, r)$, (iii) $0 \notin \cup_j \overline{D(w_j, r)}$, (iv) $w_{j+1} \in D(w_j, r)$ for $j = 1, \dots, N-1$, and (v) the traveling direction, given by the orientation, of the piece of Γ between w_j and w_{j+1}, inside $D(w_j, r)$ is from w_j to w_{j+1}.

Since $0 \notin D(w_j, r)$ we may choose a branch $\log_{\varphi_j}$ of the logarithm that is holomorphic in $D(w_j, r)$; recall that $(\log_{\varphi_j} z)' = 1/z$ where it is defined. Let Γ_j be the part of Γ between w_j and w_{j+1} inside $D(w_j, r)$. In view of Proposition 3.4 we get

$$\begin{aligned}
\int_\Gamma \frac{dz}{z} &= \sum_{j=1}^{N-1}\int_{\Gamma_j}\frac{dz}{z} = \sum_{j=1}^{N-1}\int_{\Gamma_j}(\log_{\varphi_j} z)'\,dz = \sum_{j=1}^{N-1}\log_{\varphi_j} w_{j+1} - \log_{\varphi_j} w_j \\
&= \sum_{j=1}^{N-1}\log|w_{j+1}| + i\arg_{\varphi_j} w_{j+1} - \log|w_j| - i\arg_{\varphi_j} w_j \\
&= i\sum_{j=1}^{N-1}\arg_{\varphi_j} w_{j+1} - \arg_{\varphi_j} w_j \\
&= i\sum_{j=2}^{N}\arg_{\varphi_{j-1}} w_j - \arg_{\varphi_j} w_j.
\end{aligned}$$

But $\arg_{\varphi_{j-1}} w_j - \arg_{\varphi_j} w_j$ is 2π times some integer and so part (a) of the theorem follows.

To show part (b) we notice that $\mathrm{Ind}_\Gamma(a)$ depends continuously on a in $\mathbb{C}\setminus|\Gamma|$ by Exercise 10.1. Thus, $\mathrm{Ind}_\Gamma(a)$ is a continuous integer-valued function in $\mathbb{C}\setminus|\Gamma|$. It follows that $\mathrm{Ind}_\Gamma(a)$ is constant on each connected component of $\mathbb{C}\setminus|\Gamma|$ (why?). Finally, if $|a| > 2\sup_{z\in|\Gamma|}|z|$, then $1/(1-|z/a|) < 1/2$ for all $z\in|\Gamma|$, and so

$$\big|\mathrm{Ind}_\Gamma(a)\big| = \left|\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}\right| = \frac{1}{2\pi|a|}\left|\int_\Gamma \frac{dz}{z/a-1}\right| \le \frac{1}{2\pi|a|}\frac{\ell(\Gamma)}{2}$$

by Proposition 2.7. For $|a|$ sufficiently large, $\ell(\Gamma)/(4\pi|a|) < 1$ and so, since $\mathrm{Ind}_\Gamma(a)$ is an integer, $\mathrm{Ind}_\Gamma(a) = 0$ if $|a|$ is sufficiently large. □

A cycle Γ in an open set Ω is said to be *null-homologous* in Ω if $\mathrm{Ind}_\Gamma(a) = 0$ for all $a\in\mathbb{C}\setminus\Omega$; two cycles Γ_1 and Γ_2 are said to be *homologous* in Ω if $\Gamma_1-\Gamma_2$ is null-homologous. The notion of two cycles being homologous in Ω is an equivalence relation on the set of cycles in Ω and the corresponding equivalence classes are called homology classes.

The content of the homology version of Cauchy's theorem below is that Cauchy integrals only depend on the homology class. Notice that if $\mathcal{C}$ be a null-homotopic piecewise smooth oriented closed curve in an open set Ω, then $\mathcal{C}$ is null-homologous in Ω. In fact, $1/(z-a)$ is holomorphic in $\mathbb{C}\setminus\{a\}$ and $\mathcal{C}$ is null-homotopic in $\mathbb{C}\setminus\{a\}$ so $\mathrm{Ind}_{\mathcal{C}}(a) = 0$ by the homotopy version of Cauchy's theorem.

Remark 10.6. Our definition of homology is not the standard one. However it is equivalent to the standard definition, see M. Andersson's book “Topics in complex analysis”.

Theorem 10.7 (Cauchy's theorem, homology version)**.** *Let $\Omega\subset\mathbb{C}$ be an open set and let Γ be a null-homologous cycle in Ω. Then, for any holomorphic function f in Ω,*

$$\int_\Gamma f(z)\,dz = 0.$$

Sketch of proof. [5] Let φ be a C^1-smooth function in $\mathbb{C}$ such that $\varphi(z) = 0$ if $|z|$ is sufficiently large and notice that $\mathrm{Ind}_\Gamma(z)$ is a bounded function defined almost everywhere in $\mathbb{C}$ by Theorem 10.5. In view of Exercise 3.9 we get

$$\begin{aligned} 2i\iint_{\mathbb{C}} \mathrm{Ind}_\Gamma(z)\frac{\partial\varphi}{\partial\bar z}\,dxdy &= 2i\iint_{\mathbb{C}}\frac{1}{2\pi i}\int_\Gamma\frac{dw}{w-z}\,\frac{\partial\varphi}{\partial\bar z}\,dxdy \\ &= \int_\Gamma -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\partial\varphi}{\partial\bar z}\frac{dxdy}{z-w}\,dw \\ &= \int_\Gamma \varphi(w)\,dw. \end{aligned}$$

Let f be a holomorphic function in Ω and let χ be a smooth function in $\mathbb{C}$ such that $\chi(z) = 1$ if $|z|\le R$ and $\chi(z) = 0$ if $|z|\ge R+1$, where R is to be chosen suitably. Replacing φ in the preceding computation by $f\chi$ we get

$$2i\iint_{\mathbb{C}} \mathrm{Ind}_\Gamma(z)\frac{\partial\chi}{\partial\bar z}f(z)\,dxdy = \int_\Gamma f(w)\chi(w)\,dw. \tag{10.2}$$

By Theorem 10.5, $\mathrm{Ind}_\Gamma(z) = 0$ if $|z| > R'$ for R' large enough, and so the left-hand side of (10.2) is 0 if $R > R'$. On the other hand, if R is so big that $|\Gamma|\subset D(0,R)$, then the right-hand side of (10.2) equals $\int_\Gamma f(w)\,dw$. The theorem thus follows. □

[5] This proof sketch can be made rigorous by using some distribution theory and/or integration theory.

10.3 Properties of simply connected open sets

We'll see that in simply connected open sets the theory of holomorphic functions is particularly well behaved.

Theorem 10.8. *Let $\Omega \subset \mathbb{C}$ be an open connected set. If Ω is simply connected then*

(a) $Ind_\gamma(a) = 0$ *for any parametrized piecewise smooth closed curve γ in Ω and any $a \notin \Omega$,*

(b) $\displaystyle\int_\Gamma f(z)dz = 0$ *for any cycle Γ in Ω and any holomorphic function f in Ω,*

(c) *each holomorphic function f in Ω has a holomorphic primitive in Ω, i.e., there is a holomorphic F in Ω such that $F' = f$,*

(d) *each holomorphic function f in Ω such that $f(z) \neq 0$ for all $z \in \Omega$ has a holomorphic logarithm in Ω, i.e., there is a holomorphic g in Ω such that $f(z) = e^{g(z)}$ and $g'(z) = f'(z)/f(z)$ for all $z \in \Omega$.*

We will prove that if Ω is simply connected then property (a) holds. Then we show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) without assuming that Ω is simply connected. It is also true that (d) implies that Ω is simply connected, see, e.g., Andersson. Hence, any one of the properties (a) – (d) is in fact equivalent to that Ω is simply connected.

Proof. If Ω is simply connected then (a) follows as in the paragraph preceding Remark 10.6 since any closed curve in Ω is null-homotopic.

Assume that (a) holds. Then every parametrized piecewise smooth closed curve in Ω is null-homologous and it follows that any cycle in Ω is null-homologous. Hence, (b) follows from the homology version of Cauchy's theorem.

Assume that (b) holds and let f be a holomorphic function in Ω. Fix some point $a \in \Omega$ and set, for any $z \in \Omega$, $F(z) = \int_{\gamma(a,z)} f(z)dz$, where $\gamma(a,z)$ is a piecewise smooth curve starting at a and ending at z; since (b) holds, $F(z)$ is independent of the choice of $\gamma(a,z)$ (why?). As in the proof of Morera's theorem we then see that F is holomorphic and that $F' = f$. Hence, (c) holds.

Assume that (c) holds and let f be a holomorphic function in Ω such that $f(z) \neq 0$ for all $z \in \Omega$. Then f'/f is holomorphic in Ω and there is a holomorphic function $\tilde{g}$ in Ω such that $\tilde{g}' = f'/f$. Hence,

$$\left(\frac{e^{\tilde{g}}}{f}\right)' = \frac{e^{\tilde{g}}\tilde{g}'f - e^{\tilde{g}}f'}{f^2} = \frac{e^{\tilde{g}}f' - e^{\tilde{g}}f'}{f^2} = 0,$$

and so $e^{\tilde{g}}/f = C$ for some (necessarily non-zero) constant C by Proposition 3.6. Choosing c such that $C = e^c$ we get $e^{\tilde{g}-c} = f$ and we may thus take $g := \tilde{g} - c$ to see that (d) holds. $\square$

Exercises

10.1 Let Γ be a cycle in $\mathbb{C}$, let $a \in \mathbb{C} \setminus |\Gamma|$, and let d be the distance between a and $|\Gamma|$. Show, e.g., using Proposition 2.7, that $\left|\mathrm{Ind}_\Gamma(a+h) - \mathrm{Ind}_\Gamma(a)\right| \leq |h|\dfrac{\ell(\Gamma)}{\pi d^2}$ if $|h| \leq d/2$.

11 Some mapping properties of holomorphic functions

11.1 The Maximum principle and Schwarz's lemma

The modulus of a holomorphic function cannot have a local maximum unless the function is constant. Another way to say this is that the modulus of a holomorphic function is maximal on the boundary. This is the content of the Maximum principle, a.k.a. the Maximum modulus principle. We begin with a local version.

Proposition 11.1 (Maximum principle, local version)**.** *If f is holomorphic in a disc $D(a,R)$ and $|f(z)| \leq |f(a)|$ for all $z \in D(a,R)$, then f is constant.*

Proof. By expressing the integral in Cauchy's formula using the natural parametrization of the circle $\partial D(a,r)$ we get

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})dt, \quad \text{for } r < R,$$

see Exercise 3.10. Hence, we get

$$\begin{aligned} 0 &= 2\pi|f(a)| - 2\pi|f(a)| = |f(a)|\int_0^{2\pi} dt - \left|\int_0^{2\pi} f(a+re^{it})dt\right| \qquad (11.1)\\ &\geq \int_0^{2\pi} |f(a)|dt - \int_0^{2\pi} |f(a+re^{it})|dt \\ &= \int_0^{2\pi} |f(a)| - |f(a+re^{it})|\, dt \geq 0, \end{aligned}$$

where the last step follows since $|f(a)| \geq |f(a+re^{it})|$ for all $r < R$ and all t by assumption. Thus the inequalities in this computation must be equalities. But since $|f(a)|-|f(a+re^{it})| \geq 0$, we cannot have $\int_0^{2\pi} |f(a)| - |f(a+re^{it})|\, dt = 0$ unless $|f(a)| - |f(a+re^{it})| = 0$. Hence, $|f|$ is constant in $D(a,R)$ and so f is constant in $D(a,R)$ by Proposition 3.6. □

Theorem 11.2 (Maximum principle, global version)**.** *Let $\Omega \subset \mathbb{C}$ be an open bounded set. Let f be holomorphic in Ω and assume that f extends to a continuous function on $\overline{\Omega}$. Then $|f|$ attains its maximum on $\partial\Omega$, i.e., $\sup_{z\in\overline{\Omega}} |f(z)| = \sup_{z\in\partial\Omega} |f(z)|$.*

Proof. Since $|f|$ is a continuous function on the compact set $\overline{\Omega}$ there is an $a \in \overline{\Omega}$ such that $|f(a)| = \sup_{z\in\overline{\Omega}} |f(z)|$. If $a \in \partial\Omega = \overline{\Omega}\setminus\Omega$ we are done, so assume that $a \in \Omega$. Since Ω is open there is a disc $D(a,R) \subset \Omega$ and so, by the local version of the Maximum principle, f is constant in $D(a,R)$. But then, by Corollary 9.8, f is constant in the connected component Ω' of Ω that contains a. Thus, $f(z) = f(a)$ for any boundary point of Ω'; such points are also boundary points of Ω and so there are boundary points of Ω where $|f|$ attains its maximum. □

One application of the Maximum principle is Schwarz's lemma, which in particular will enable us to determine all bijective holomorphic maps from the unit disc to itself.

Theorem 11.3 (Schwarz's lemma)**.** *Let f be holomorphic in the disc $D(0,R)$ and assume that $f(0) = 0$ and that $|f(z)| \leq M$ for all $z \in D(0,R)$. Then $|f(z)| \leq M|z|/R$ for all $z \in D(0,R)$ and $|f'(0)| \leq M/R$. Moreover, if $|f(z)| = M|z|/R$ for some $z \in D(0,R)\setminus\{0\}$ or if $|f'(0)| = M/R$, then $f(z) = z\frac{M}{R}e^{i\theta}$ for some constant $\theta \in \mathbb{R}$.*

Proof. We may assume that $M = R = 1$; otherwise consider the function $\tilde{f}(z) = f(zR)/M$.

Since $f(0) = 0$ we can write $f(z) = zg(z)$ for some g holomorphic in $D(0,1)$, cf. Proposition 9.5. Notice that $f'(0) = g(0)$. For any $r < 1$, g is continuous on $\overline{D(0,r)}$ and $|g(z)| = |f(z)|/|z| \leq 1/r$ if $|z| = r$. Thus, by the Maximum principle, $|g(z)| \leq 1/r$ in $D(0,r)$. Letting $r \to 1^-$ it follows that $|g(z)| \leq 1$ in $D(0,1)$, and so $|f(z)| \leq |z|$ in $D(0,1)$.

Assume that $|f(z)| = |z|$ for some $z \in D(0,1) \setminus \{0\}$ or that $|f'(0)| = 1$. By the first part of the proof, $|g|$ then has a maximum $= 1$ in $D(0,1)$ and so by the local version of the Maximum principle g is a constant, which has to have modulus 1. Thus, $g(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and the last part of the theorem follows. □

Notice that the hypothesis of Schwarz's lemma is that f is a holomorphic function from the disc $D(0,R)$ to the disc $D(0,M)$ such that $f(0) = 0$. If $f(0) = a \neq 0$ we can still get some information from Schwarz's lemma. Set $\alpha = a/M \in D(0,1)$ and let

$$\phi_\alpha(z) = \frac{z - \alpha}{\bar{\alpha}z - 1}. \tag{11.2}$$

One can check that ϕ_α is a bijective map from $D(0,1)$ to itself (Exercise 11.3). The function $h(z) := \phi_\alpha\big(f(z)/M\big)$ then is a holomorphic map from $D(0,R)$ to $D(0,1)$ such that $h(0) = 0$. Schwarz's lemma thus gives $|h(z)| \leq |z|/R$, i.e., $|\phi_\alpha\big(f(z)/M\big)| \leq |z|/R$, which at least says something about f.

Let us also show how Schwarz's lemma can be used to prove that any bijective holomorphic maps from the unit disc to itself, i.e., a holomorphic automorphism of $D(0,1)$, is of the form ϕ_α for some $\alpha \in D(0,1)$, up to a multiplicative constant of modulus 1. Let f be a holomorphic automorphism of $D(0,1)$. Suppose first that $f(0) = 0$. Then Schwarz's lemma gives $|f(z)| \leq |z|$ for all $z \in D(0,1)$. On the other hand, Schwarz's lemma applied to f^{-1} gives $|f^{-1}(z)| \leq |z|$ for all $z \in D(0,1)$. Hence, $|z| = |f^{-1}(f(z))| \leq |f(z)| \leq |z|$ for all $z \in D(0,1)$ and so the inequalities must be equalities. From Schwarz's lemma again it follows that $f(z) = e^{i\theta}z = e^{i(\pi+\theta)}\phi_0(z)$ for some $\theta \in \mathbb{R}$. If $f(0) = a \neq 0$, then $\phi_a \circ f$ is an automorphism of $D(0,1)$ that maps 0 to 0. From what we have just seen, $\phi_a \circ f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ and so, since $\phi_a^{-1} = \phi_a$ (Exercise 11.3),

$$f(z) = \phi_a(e^{i\theta}z) = e^{i\theta}\phi_{e^{-i\theta}a}(z);$$

the last equality is a simple exercise.

11.2 Three basic mapping theorems and the Riemann mapping theorem

Recall from Proposition 3.6 and Exercise 3.7 that if the image of a holomorphic function is contained in a circle or a line then the function must be constant. This suggests that non-constant holomorphic functions cannot have "thin" images. In fact, the next result says that the image of a non-constant holomorphic function necessarily is open.

Theorem 11.4. *Let f be holomorphic in an open connected set $\Omega \subset \mathbb{C}$. Then either f is constant or $f(\Omega)$ is open.*

Proof. Assume that f is not constant. Let $a \in \Omega$ be an arbitrary point; we will show that there is a disc centered at $f(a)$ contained in $f(\Omega)$.

We may assume that $f(a) = 0$ (possibly after replacing f by $f - f(a)$). Since f is not constant, a is an isolated zero of f by the Identity theorem II and so we may choose a disc $D(a,r) \subset \Omega$ such that $f(z) \neq 0$ for $z \in \partial D(a,r)$. Let $\epsilon := \inf_{z \in \partial D(a,r)} |f(z)|$; since $|f|$ is continuous and non-zero on the compact set $\partial D(a,r)$, we have $\epsilon > 0$. Consider the disc $D(0,\epsilon)$ and let $w \in D(0,\epsilon)$. Let g be the constant function $g(z) = -w$. For $z \in \partial\Omega$ we have $|f(z)| \geq$

$\epsilon > |w| = |g(z)|$ and so by Rouche's theorem f and $f+g$ have the number of zeros in $D(a,r)$. Thus, since f has a zero in $D(a,r)$, there is a $b \in D(a,r)$ such that $f(b)+g(b)=0$. Hence, $f(b)=w$ and it follows that $D(0,\epsilon) \subset f(D(a,r)) \subset f(\Omega)$. □

With some variation of this proof we also get

Theorem 11.5. *If f is an injective holomorphic function defined in some open set $\Omega \subset \mathbb{C}$, then f is conformal in Ω, i.e., $f'(z) \neq 0$ for all $z \in \Omega$.*

Proof. Assume, to get a contradiction, that there is a point $a \in \Omega$ such that $f'(a)=0$. We may assume that $f(a)=0$ (possibly after replacing f by $f - f(a)$). Since f is injective, f cannot be identically zero in any disc centered at a and so, by the Identity theorem I, a is a zero of some finite order m; notice that then a is a zero of f' of order $m-1 \geq 1$. Thus, by Proposition 9.5, there is a disc $D(a,r) \subset \Omega$ such that a is the only zero of f and f' in $\overline{D(a,r)}$. Set $\epsilon := \inf_{z \in \partial D(a,r)} |f(z)|$. As in the previous proof, $\epsilon > 0$ and, for any $w \in D(0,\epsilon)$, $f(z)$ and $f(z)-w$ have the same number of zeros in $D(a,r)$. Choose $w_0 \in D(0,\epsilon) \setminus \{0\}$ and notice that $f(a)-w_0 \neq 0$ since f is injective. Since f has a zero of order $m \geq 2$ at a, $f(z)-w_0$ must have at least two zeros in $D(a,r)$ taking multiplicity into account. Now, $f(z)-w_0$ cannot have two distinct zeros since f is injective and so $f(z)-w_0$ must have a zero of multiplicity at least two at some $b \in D(a,r) \setminus \{a\}$. But then $f'(b)=0$, which contradicts that a is the only zero of f' in $D(a,r)$. □

Theorem 11.6 (Inverse function theorem)**.** *Let f be an injective holomorphic function defined in some open set $\Omega \subset \mathbb{C}$. Then f has a holomorphic inverse $f^{-1}\colon f(\Omega) \to \Omega$.*

Proof. Since f is injective it is set-theoretic nonsense that there is a unique inverse $f^{-1}\colon f(\Omega) \to \Omega$; we'll need to show that f^{-1} is holomorphic.

Let $w_0 \in f(\Omega)$ and let $z_0 = f^{-1}(w_0)$. Since f is injective $f'(z_0) \neq 0$ by Theorem 11.5 and so the differential of f at z_0 is non-zero, cf. (2.8). It thus follows from the Inverse function theorem of calculus that f^{-1} is C^1-smooth close to w_0. To show that f^{-1} is complex differentiable at w_0 note that if w_j is a sequence converging to w_0 then $z_j := f^{-1}(w_j)$ converges to z_0 by continuity. Hence,

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{f^{-1}(f(z)) - f^{-1}(f(z_0))}{f(z) - f(z_0)} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)},$$

where we compute the limit by considering sequences $\{w_j\}$ such that $w_j \neq w_0$; this ensures that $f(z_j) - f(z_0) \neq 0$ since f is injective. □

If f is an injective holomorphic function defined in some open Ω, f^{-1} thus is a holomorphic function in $f(\Omega)$, which is open by Theorem 11.4, at least if Ω is connected. Moreover, by Theorem 11.5, both f and f^{-1} are conformal. We say that two open sets are *conformally equivalent* if there is a bijective holomorphic map between them. In Chapter 7 we indicated the importance of finding conformal mappings from complicated open sets to simpler ones and we also looked at some techniques to construct such maps. Given a quite general open set it is very hard to construct an explicit conformal mapping onto some substantially simpler one, but, in fact, theoretically it is usually possible. The first and main result in this direction is the Riemann mapping theorem.

Theorem 11.7 (Riemann mapping theorem)**.** *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be an open simply connected set. Then Ω is conformally equivalent to the unit disc.*

A proof of this is beyond the scope of these notes but can be found in, e.g., M. Andersson's book "Topics in complex analysis". However, in Chapter 14 below we will make it physically reasonable that the Riemann mapping theorem holds.

Notice that $\mathbb{C}$ cannot be conformally equivalent to the unit disc because if it where, then there would be a bijective holomorphic map $f\colon \mathbb{C} \to D(0,1)$ and this is not possible by Liouville's theorem.

In view of the Riemann mapping theorem the hypothesis in Schwarz's lemma becomes less restrictive than it might look. To be more specific, let f be a holomorphic function in an open simply connected set $\Omega \neq \mathbb{C}$ and assume that $f(\Omega) \neq \mathbb{C}$. By the Riemann mapping theorem there are bijective holomorphic maps $\varphi\colon D(0,1) \to \Omega$ and $\psi\colon f(\Omega) \to D(0,1)$ and so $\psi \circ f \circ \varphi$ is a map from the unit disc to itself. By composing further with suitable ϕ_α's, see (11.2), we may assume that 0 is mapped to 0. Then the hypothesis of Schwarz's lemma is satisfied and one might get some information about f. One illustration of this is outlined in Exercise 11.2.

Exercises

11.1 Let f be a holomorphic function in $\mathbb{C}$ such that $f(z) \in \mathbb{R}$ for all $z \in \partial D(0,1)$.

(a) Show that $|e^{\pm i f(z)}| = 1$ for z with $|z| = 1$.

(b) Use the Maximum principle to show that $f(z) \in \mathbb{R}$ for all $z \in D(0,1)$ and conclude that f is a constant function.

11.2 Let Π^+ be the upper half-plane, let $a \in \Pi^+$, and let $f\colon \Pi^+ \to \Pi^+$ be a holomorphic map. Show that $\left|\dfrac{f(z) - f(a)}{f(z) - \overline{f(a)}}\right| \leq \left|\dfrac{z-a}{z-\bar{a}}\right|$ for all $z \in \Pi^+$. (Hint: Consider the composition $\phi_2 \circ f \circ \phi_1^{-1}$ where $\phi_1(z) = (z-a)/(z-\bar{a})$ and $\phi_2(z) = (z - f(a))/(z - \overline{f(a)})$.)

11.3 (a) Show that $|z - \alpha| < |\bar{\alpha} z - 1|$ if and only if $|z|^2(1 - |\alpha|^2) < 1 - |\alpha|^2$, and conclude that $|\phi_\alpha(z)| < 1$ if $z, \alpha \in D(0,1)$.

(b) Show that $w = \phi_\alpha(z)$ if and only if $z = \phi_\alpha(w)$.

12 Singularities and Cauchy's residue theorem

By singularities we here mean isolated points where a holomorphic function is not defined. We will see that a holomorphic function is represented by a Laurent series in a neighborhood of a singularity; a Laurent series is a power series that also contains negative powers. Laurent series expansions allow us to classify the singularities of holomorphic functions.

Cauchy's residue theorem is a generalization of Cauchy's theorem/formula that applies to holomorphic functions with singularities.

12.1 Laurent series and classification of singularities

Let A be the annulus $\{z;\, r<|z|<R\}$ and let f be a holomorphic function in an open set containing $\overline{A}$. For any $z\in A$ we have by Cauchy's formula that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial A}\frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\int_{|w|=R}\frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)\,dw}{w-z} \\ &= \frac{1}{2\pi i}\int_{|w|=R}\frac{f(w)}{w}\frac{dw}{1-z/w} + \frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)}{z}\frac{dw}{1-w/z}. \end{aligned} \tag{12.1}$$

Since $|z/w|<1$ if $|w|=R$ and $|w/z|<1$ if $|w|=r$ we have $1/(1-z/w)=\sum_{k=0}^{\infty}(z/w)^k$ if $|w|=R$ and $1/(1-w/z)=\sum_{k=0}^{\infty}(w/z)^k$ if $|w|=r$. As in the proof of Theorem 4.7 one shows

$$\int_{|w|=R}\frac{f(w)}{w}\frac{dw}{1-z/w} = \sum_{k=0}^{\infty}\int_{|w|=R}\frac{f(w)}{w}\left(\frac{z}{w}\right)^k dw \quad \text{and}$$

$$\int_{|w|=r}\frac{f(w)}{z}\frac{dw}{1-w/z} = \sum_{k=0}^{\infty}\int_{|w|=r}\frac{f(w)}{z}\left(\frac{w}{z}\right)^k dw,$$

and so, by (12.1),

$$f(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty} z^k\int_{|w|=R}\frac{f(w)}{w^{k+1}}dw + \frac{1}{2\pi i}\sum_{k=0}^{\infty} z^{-k-1}\int_{|w|=r} f(w)w^k dw.$$

Both of these series converge if $r<|z|<R$ and so it follows from Lemma 4.4 that the first series is absolutely convergent if $|z|<R$ and the second one is absolutely convergent if $|z|>r$. Setting $c_k := 1/(2\pi i)\int_{\mathcal{C}} f(w)/w^{k+1}\,dw$, where $\mathcal{C}$ is any curve such that $\mathcal{C}-\partial D(0,R)$ is null-homologous in A, we have $f(z)=\sum_{-\infty}^{\infty}c_k z^k$ and the series converges absolutely in A. This is the Laurent series of f (centered at 0).

Theorem 12.1 (Laurent's theorem)**.** *Let $A=\{z\in\mathbb{C};\, r<|z-a|<R\}$ and let f be a holomorphic function in an open set containing $\overline{A}$. Then there is a unique series $\sum_{-\infty}^{\infty}c_k(z-a)^k$ converging absolutely in A such that*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z-a)^k$$

for $z\in A$. The coefficients are given by

$$c_k = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(w)\,dw}{(w-a)^{k+1}}, \tag{12.2}$$

where $\mathcal{C}$ is any curve such that $\mathcal{C}-\partial D(a,\varrho)$, $r<\varrho<R$, is null-homologous in A.

Proof. We may assume that $a = 0$ and we have already showed that if c_k is defined by (12.2) (with $a = 0$) then $f(z) = \sum_{-\infty}^{\infty} c_k z^k$ for $z \in A$ and the series is absolutely convergent in A. It remains to show uniqueness.

Let $\sum_{-\infty}^{\infty} d_k z^k$ be absolutely convergent in A and assume that $f(z) = \sum_{-\infty}^{\infty} d_k z^k$ for $z \in A$. To see that $d_k = c_k$ for all k we notice that if $r < \varrho < R$, then

$$2\pi i c_\ell = \int_{|w|=\varrho} \frac{f(w)\,dw}{w^{\ell+1}} = \sum_{k=-\infty}^{\infty} d_k \int_{|w|=\varrho} \frac{w^k}{w^{\ell+1}} dw = 2\pi i d_\ell; \tag{12.3}$$

the last equality follows from Exercise 2.9 and an argument showing that it is allowed to interchange the order of integration and summation in the second equality is outlined in Exercise 12.1. □

To find a Laurent series explicitly in a concrete example it is often easier to use some ad hoc method instead of trying to compute the coefficients by (12.2). If one somehow can cook up a Laurent-type series representing a given function then it has to be the Laurent series by uniqueness of Laurent series.

Example 12.2. Find the Laurent series of $f(z) = \dfrac{1}{z(z-1)}$ centered at 1.
Solution: One way to solve this problem is first to split f into partial fractions,

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}, \tag{12.4}$$

and then use the formula for a geometric series to write

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{k=0}^{\infty} (-1)^k (z-1)^k,$$

which converges in $D(1,1) \setminus \{1\}$. Hence, the Laurent series of f in $D(1,1) \setminus \{1\}$ is

$$\frac{1}{z-1} - \sum_{k=0}^{\infty} (-1)^k (z-1)^k.$$

One could also use the formula for a geometric series directly to get

$$\frac{1}{z(z-1)} = \frac{1}{z-1} \sum_{k=0}^{\infty} (-1)^k (z-1)^k = \sum_{k=-1}^{\infty} (-1)^{k+1} (z-1)^k,$$

which is the same as we got before.

The part of the Laurent series containing negative powers is called *the principal part.* In the example above the principal part is $1/(z-1)$.

Let f be holomorphic in the annulus $A = \{z;\, r < |z-a| < R\}$. Then its Laurent series $\sum_{-\infty}^{\infty} c_k (z-a)^k$ converges to f in $\{z;\, r+\epsilon < |z-a| < R-\epsilon\}$ for any $\epsilon > 0$ by Laurent's theorem, and so it converges to f in A. If the principal part of the Laurent series vanishes, i.e., if $c_k = 0$ for $k < 0$, then f extends to a holomorphic function in the disc $D(a,R)$. In fact, if the Laurent series converges in $\{z;\, r < |z-a| < R\}$ and has no negative powers of $z-a$ then the series converges in $D(a,R)$ by Lemma 4.4 and thus gives the holomorphic extension.

Let $r = 0$, so that f has a singularity at a. If $c_k = 0$ for $k < 0$, then, as we have just seen, f extends across a, and we say that f has a *removable singularity* at a. If there is an $m > 0$ such that $c_{-m} \neq 0$ and $c_k = 0$ for $k < -m$, then we say that f has a *pole* of order m at a. In the example above the function has a pole of order 1 at 1. If $c_k \neq 0$ for infinitely many negative k's, then we say that f has an *essential singularity* at a.

Proposition 12.3. *Let f be holomorphic in the punctured disc $D(a,R)\setminus\{a\}$ and assume that there is a constant $M>0$ such that $|f(z)|\le M$ for all $z\in D(a,R)\setminus\{a\}$. Then f has a removable singularity at a.*

Proof. Let c_k, $k\in\mathbb{Z}$, be the coefficients in the Laurent series of f centered at a and let $0<\varrho<R$. By Laurent's theorem,

$$|c_k| = \left|\frac{1}{2\pi i}\int_{|z-a|=\varrho}\frac{f(z)\,dz}{(z-a)^{k+1}}\right| \le \frac{1}{2\pi}\sup_{|z-a|=\varrho}\left|\frac{f(z)}{|z-a|^{k+1}}\right| 2\pi\varrho \le \frac{M}{\varrho^k}.$$

If $k<0$, then $M/\varrho^k\to 0$ as $\varrho\to 0$ and so $c_k=0$ if $k<0$. Hence, f has a removable singularity at a. □

A function f holomorphic in an open set Ω except for poles at isolated points $a_j\in\Omega$ is said to be *meromorphic* in Ω. Alternatively, f is meromorphic in Ω if for each $a\in\Omega$ either f or $1/f$ is holomorphic in $D(a,r)$ for some $r>0$. This follows in particular from the next result.

Theorem 12.4. *Let f be holomorphic in a puntured disc $D(a,R)\setminus\{a\}$. Then f has a pole of order m at a if and only if $1/f$ is holomorphic in some disc $D(a,r)$ and has a zero of order m at a.*

Proof. Assume first that f has a pole of order m at a. Then the Laurent series looks like

$$\sum_{k=-m}^{\infty} c_k(z-a)^k = (z-a)^{-m}\sum_{k=0}^{\infty} c_{k-m}(z-a)^k,$$

and converges in $D(a,R)\setminus\{a\}$ It then follows from Lemma 4.4 that $\sum_{k=0}^{\infty} c_{k-m}(z-a)^k =: g(z)$ converges in $D(a,R)$ and thus g is holomorphic in $D(a,R)$. Moreover, $g(a)=c_{-m}\neq 0$ and so $1/g$ is holomorphic in some disc $D(a,r)$ (why?). Hence, $1/f(z)=(z-a)/g(z)$ is holomorphic in $D(a,r)$ and has a zero of order m at a by Proposition 9.5.

Showing the converse statement essentially amounts to do the above reasoning backwards. Assume that $1/f$ is holomorphic in some disc centered at a and that it has a zero of order m there. Then, by Proposition 9.5, $1/f(z)=(z-a)^m\tilde{g}(z)$ where $\tilde{g}$ is holomorphic and $\tilde{g}(a)\neq 0$. Hence, $g(z):=1/\tilde{g}(z)$ is holomorphic in a disc centered at a. Let $\sum_{k=0}^{\infty} d_k(z-a)^k$ be the Taylor series of g and notice that $d_0=g(a)\neq 0$. We get

$$f(z)=\frac{g(z)}{(z-a)^m}=\frac{1}{(z-a)^m}\sum_{k=0}^{\infty} d_k(z-a)^k=\sum_{k=-m}^{\infty} d_{m+k}(z-a)^k,$$

which by uniqueness has to be the Laurent series of f. Thus f has a pole of order m at a. □

In view of Proposition 9.5 we now also get two more characterization of f having a pole of order m at a namely 1) that $f(z)=g(z)/(z-a)^m$ for some holomorphic g with $g(a)\neq 0$ and 2) that $\lim_{z\to a}(z-a)^m f(z)$ exists and is non-zero.

We conclude this section by by saying a few words about singularities at ∞. If f is holomorphic in $\{z;\,|z|>R\}$ for some $R>0$ we say that f has a singularity at ∞. Set $\tilde{f}(w)=f(1/w)$; then $\tilde{f}$ is holomorphic in the punctured disc $D(0,1/R)$. We say that f has a removable singularity at ∞, a pole of order m at ∞, and an essential singularity at ∞ if $\tilde{f}$ has a removable singularity, a pole of order m, and an essential singularity, respectively, at 0.

Example 12.5. If f is a polynomial of degree n, then f has a pole of order n at ∞. In fact, if $f(z)=\sum_{k=0}^{n} c_k z^k$ then $f(1/w)=(1/w^n)\sum_{k=0}^{n} c_k w^{n-k}$ and $\sum_{k=0}^{n} c_k w^{n-k}$ is holomorphic and non-zero at $w=0$.

By the Fundamental theorem of algebra, a polynomial thus has the same number of zeros and poles on the Riemann sphere. Something more general is in fact true; any meromorphic function on the Riemann sphere has the same number of zeros and poles. We will not prove this statement in these notes.

12.2 Residue theorems

Let f be holomorphic in a punctured disc $D(a,R)$ and let $\sum_{-\infty}^{\infty} c_k(z-a)^k$ be the Laurent series. The coefficient c_{-1} is of particular importance; it is called the *residue* of f at a and denoted $\mathrm{Res}(f;a)$. Its importance stems from the fact that

$$\int_{\partial D(a,\varrho)} f(z)\,dz = 2\pi i \mathrm{Res}\,(f;a). \tag{12.5}$$

To see this we compute as in the uniqueness part of the proof of Laurent's theorem, cf. (12.3);

$$\int_{\partial D(a,\varrho)} f(z)\,dz = \int_{\partial D(a,\varrho)} \sum_{k=-\infty}^{\infty} c_k(z-a)^k\,dz = \sum_{k=-\infty}^{\infty} c_k \int_{\partial D(a,\varrho)} (z-a)^k\,dz = 2\pi i c_{-1}.$$

Theorem 12.6 (Cauchy's residue theorem)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let f be holomorphic in $\Omega \setminus \{a_1,\dots,a_N\}$. Let $\omega \subset \Omega$ be an open subset such that $a_j \in \omega$ for all j, $\overline{\omega} \subset \Omega$, and $\partial\omega$ is a finite number of piecewise smooth closed curves. Then*

$$\int_{\partial\omega} f(z)\,dz = 2\pi i \sum_{j=1}^{N} Res(f;a_j).$$

Bevis. Choose $r > 0$ such that $\overline{D(a_j,r)} \subset \omega$ for all j. Then f is holomorphic in an open set containing $\overline{\omega} \setminus \cup_j D(a_j,r)$ and so it follows from Cauchy's theorem that

$$\int_{\partial\omega} f(z)\,dz = \sum_{j=1}^{N} \int_{\partial D(a_j,r)} f(z)\,dz.$$

But by (12.5) the right-hand side equals $2\pi i \sum_{j=1}^{N} \mathrm{Res}\,(f;a_j)$ and we are done. □

Example 12.7. Compute the integral $\displaystyle\int_{\partial D(0,2)} \frac{dz}{z(z-1)}$.

Solution: From (12.4) we read off that $1/(z(z-a))$ has poles of order 1 at $z=0$ and $z=1$, which are in $D(0,2)$, and that the residues at these points are -1 and 1, respectively (why?). By the residue theorem the integral thus equals $2\pi i(-1+1)=0$.

Theorem 12.6 is sufficient to solve most basic problems but we also give a jazzed-up version.

Theorem 12.8. *Let $\Omega \subset \mathbb{C}$ be an open set and let f be holomorphic in $\Omega \setminus \{a_1,\dots,a_N\}$. If Γ is a null-homologous cycle in Ω such that $a_j \notin |\Gamma|$ for all j, then*

$$\int_{\Gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{N} Ind_{\Gamma}(a_j) Res(f;a_j).$$

Proof. Choose $r > 0$ such that the discs $\overline{D(a_j,r)}$, $j = 1,\dots,N$, are pairwise disjoint and $\overline{D(a_j,r)} \subset \Omega$ and $\overline{D(a_j,r)} \cap |\Gamma| = \emptyset$ for all j. Consider the cycle $\gamma = \sum_j \mathrm{Ind}_{\Gamma}(a_j)\partial D(a_j,r)$.

Since $\overline{D(a_j,r)} \subset \Omega$ the winding number $\mathrm{Ind}_\gamma(z)$ is 0 for all $z \notin \Omega$. Hence, the cycle $\tilde\Gamma := \Gamma - \gamma$ is null-homologous in Ω. Since also $\mathrm{Ind}_{\tilde\Gamma}(a_j) = \mathrm{Ind}_\Gamma(a_j) - \mathrm{Ind}_\gamma(a_j) = 0$ by the choice of γ, the cycle $\tilde\Gamma$ is in fact null-homologous in $\Omega\setminus\{a_1,\dots,a_N\}$. Since f is holomorphic in $\Omega\setminus\{a_1,\dots,a_N\}$ it follows from the homology version of Cauchy's theorem that $\int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz$, and so

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^N \mathrm{Ind}_\Gamma(a_j)\int_{\partial D(a_j,r)} f(z)\,dz = 2\pi i \sum_{j=1}^N \mathrm{Ind}_\Gamma(a_j)\mathrm{Res}(f;a_j).$$

□

In order for the Residue theorem to be useful we need ways of computing residues efficiently; we list a few here.

Assume first that f has a pole of order $\le m$ at a. Then $g(z) = (z-a)^m f(z)$ is holomorphic by the comment following the proof of Theorem 12.4; let $\sum_{k=0}^\infty c_k(z-a)^k$ be the Taylor series of g. The Laurent series of f thus is $\sum_{k=0}^\infty c_k(z-a)^{k-m}$ and we read off that $\mathrm{Res}\,(f;a) = c_{m-1} = g^{(m-1)}(a)/(m-1)!$. We get the following computation rules.

Residue computation 1a: If f has a pole of order $\le m$ at a then $\mathrm{Res}\,(f;a) = \dfrac{g^{(m-1)}(a)}{(m-1)!}$ where $g(z) = (z-a)^m f(z)$.

Residue computation 1b: If f is of the form $f(z) = (z-a)^{-m}\sum_{k=0}^\infty c_k(z-a)^k$, then $\mathrm{Res}\,(f;a) = c_{m-1}$.

Residue computation 1c: If f has a pole of order 1 at a then $\mathrm{Res}\,(f;a) = \lim_{z\to a}(z-a)f(z)$.

Assume now that f is of the form $f(z) = g(z)/h(z)$, where h has a zero of order 1 at a. Then $h(z) = (z-a)\tilde h(z)$ where $\tilde h(a) \neq 0$ in view of Proposition 9.5; a straightforward computation (do it!) shows that $\tilde h(a) = h'(a)$. Hence, $f(z) = (g(z)/\tilde h(z))/(z-a)$ and it follows by Residue computation 1c that $\mathrm{Res}\,(f;a) = g(a)/\tilde h(a) = g(a)/h'(a)$. We thus have

Residue computation 2: If f is of the form $f(z) = g(z)/h(z)$, where h has a zero of order 1 at a, then $\mathrm{Res}\,(f;a) = g(a)/\tilde h(a) = g(a)/h'(a)$.

Exercises

12.1 Let $\sum_{-\infty}^\infty d_k z^k$ be absolutely convergent in $A = \{z;\, r < |z| < R\}$, let $r < \varrho < R$, and choose u and v such that $r < u < \varrho < v < R$.

(a) Show that there is a constant C such that $|d_k|v^k \le C$ and $|d_k|u^k \le C$ for all $k \in \mathbb{Z}$.

(b) Show that

$$\sup_{|w|=\varrho}\left|\sum_{k=N+1}^\infty d_k w^k\right| \le C\left(\frac{\varrho}{v}\right)^{N+1}\frac{1}{1-\varrho/v},$$

$$\sup_{|w|=\varrho}\left|\sum_{k=N+1}^\infty d_{-k}\frac{1}{w^k}\right| \le C\left(\frac{u}{\varrho}\right)^{N+1}\frac{1}{1-u/\varrho}.$$

(c) Show that

$$\int_{|w|=\varrho}\sum_{k=0}^\infty d_k w^k\frac{dw}{w^{\ell+1}} - \sum_{k=0}^N\int_{|w|=\varrho} d_k w^k\frac{dw}{w^{\ell+1}} \to 0 \text{ as } N\to\infty,$$

$$\int_{|w|=\varrho}\sum_{k=1}^\infty \frac{d_{-k}}{w^k}\frac{dw}{w^{\ell+1}} - \sum_{k=1}^N\int_{|w|=\varrho} \frac{d_{-k}}{w^k}\frac{dw}{w^{\ell+1}} \to 0 \text{ as } N\to\infty,$$

and conclude that the second equality in (12.3) is valid.

12.2 Let f be holomorphic in $\mathbb{C}$, let p be a polynomial of degree d with d distinct zeros $a_1, \ldots, a_d$, and choose $R > 0$ such that $D(0, R)$ contains all the zeros of p. The purpose of this exercise is to show that for any $z \in D(0, R)$

$$f(z) = \frac{p(z)}{2\pi i} \int_{\partial D(0,R)} \frac{f(w)}{(w-z)p(w)}\, dw + \sum_{k=1}^{d} \frac{p(z)}{z - a_j} \frac{f(a_j)}{p'(a_j)}. \tag{12.6}$$

This is an example of a division formula. The reason for the name is that it shows that f is holomorphically divisible by p if $f(a_j) = 0$ for all j. In fact, if $f(a_j) = 0$ for all j, then we see that $f(z) = p(z)q(z)$, where q is given by the integral on the right-hand side, which is a holomorphic function in $D(0, R)$ (why?).

(a) Fix z and set $H_z(w) = (p(z) - p(w))/(z - w)$. Show that $H_z(w)$ has a removable singularity at $w = z$ and thus defines a holomorphic function in $\mathbb{C}$.

(b) Use the Residue theorem to show that

$$\frac{1}{2\pi i} \int_{\partial D(0,R)} \frac{H_z(w) f(w)}{p(w)}\, dw = \sum_{k=1}^{d} \frac{p(z)}{z - a_j} \frac{f(a_j)}{p'(a_j)}.$$

(c) Use Cauchy's formula to show that

$$\frac{1}{2\pi i} \int_{\partial D(0,R)} \frac{H_z(w) f(w)}{p(w)}\, dw = f(z) - \frac{p(z)}{2\pi i} \int_{\partial D(0,R)} \frac{f(w)}{(w-z)p(w)}\, dw,$$

and conclude that (12.6) holds.

13 Calculating real integrals using complex analysis

We consider a few examples illustrating some techniques to compute certain real integrals by using complex analysis methods, in particular the Residue theorem.

Example 13.1. Compute the integral $\displaystyle\int_0^{2\pi} \frac{dt}{1+8\cos^2 t}$.

The idea is to rewrite the integral as a contour integral and then use the Residue theorem. A natural first attempt is to try to rewrite the integral as an integral over $\partial D(0,1)$: Notice that $z=e^{it}$, $0 \le t < 2\pi$, is a parametrization of $\partial D(0,1)$ and that then $dz = ie^{it}dt = izdt$ and $\cos t = (e^{it}+e^{-it})/2 = (z+1/z)/2$. Hence,

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{1+8\cos^2 t} &= \int_{\partial D(0,1)} \frac{dz/iz}{1+8((z+1/z)/2)^2} = \int_{\partial D(0,1)} \frac{dz/iz}{1+2(z^2+2+1/z^2)} \\ &= -i\int_{\partial D(0,1)} \frac{zdz}{2z^4+5z^2+2}. \end{aligned} \tag{13.1}$$

The denominator is a degree 2 expression in z^2 and may be factorized as $2z^4+5z^2+2 = (2z^2+1)(z^2+2)$. It follows that the denominator has zeros of order 1 at $\pm i/\sqrt{2}$ and $\pm i\sqrt{2}$. Of these only $\pm i/\sqrt{2}$ are in $D(0,1)$ and the residue of $z/((2z^2+1)(z^2+2)) =: g(z)/(2z^2+1)$ at these points are

$$\frac{g(z)}{4z}\Big|_{z=\pm i/\sqrt{2}} = \frac{1}{4(z^2+2)}\Big|_{z=\pm i/\sqrt{2}} = \frac{1}{6},$$

by Residue computation 2. Hence, by (13.1) and the Residue theorem,

$$\int_0^{2\pi} \frac{dt}{1+8\cos^2 t} = -i2\pi i(1/6+1/6) = 2\pi/3.$$

Example 13.2. Compute the integral $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4}$.

This is an integral over the interval $[0,\infty)$ which does not bound any open set. To use the Residue theorem we need to close up the curve in an appropriate way. A first thing to notice is that $\int_0^\infty \frac{dx}{1+x^4} = \lim_{R\to\infty}\int_0^R \frac{dx}{1+x^4}$, and the interval $[0,R]$ is at least finite. We also notice that the integrand is an even function so that

$$\int_0^R \frac{dx}{1+x^4} = \frac{1}{2}\int_{-R}^R \frac{dx}{1+x^4}.$$

Denote the interval $[-R,R]$ by γ_R and let Γ_R be the upper half of the circle $\{z;\, |z|=R\}$ oriented counterclockwise. Then $\gamma_R+\Gamma_R$ is the boundary of the upper half of the disc $D(0,R)$ and we should be able to compute $\int_{\gamma_R+\Gamma_R} dz/(1+z^4)$ by the Residue theorem. On the other hand,

$$\int_{\gamma_R+\Gamma_R} \frac{dz}{1+z^4} = \int_{-R}^R \frac{dx}{1+x^4} + \int_{\Gamma_R} \frac{dz}{1+z^4}, \tag{13.2}$$

and so, if we also can compute the second integral on the right-hand side, we will get information about the integral of interest.

We begin by computing the left-hand side of (13.2). Since $1+z^4 = (z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})$ we see that, in the upper half of the disc $D(0,R)$, $f(z) := 1/(1+z^4)$ has poles of order 1 at $e^{i\pi/4}$ and $e^{i3\pi/4}$ (at least if $R>1$). The residues at these points can be computed using, e.g., Residue computation 2:

$$\mathrm{Res}(f; e^{i\pi/4}) = \frac{1}{4z^3}\Big|_{z=e^{i\pi/4}} = \cdots = -\frac{1+i}{4\sqrt{2}},$$

$$\operatorname{Res}(f; e^{i3\pi/4}) = \frac{1}{4z^3}\Big|_{z=e^{i3\pi/4}} = \cdots = \frac{1-i}{4\sqrt{2}}.$$

Hence, for any $R > 1$ we have

$$\int_{\gamma_R+\Gamma_R} \frac{dz}{1+z^4} = 2\pi i(\operatorname{Res}(f; e^{i\pi/4}) + \operatorname{Res}(f; e^{i3\pi/4})) = \cdots = \frac{\pi}{\sqrt{2}}.$$

The second integral on the right-hand side of (13.2) actually goes to 0 as $R \to \infty$. To see this we use Proposition 2.7 and the Reverse triangle inequality:

$$\left|\int_{\Gamma_R} \frac{dz}{1+z^4}\right| \leq \sup_{z\in\Gamma_R} \frac{1}{|1+z^4|} \cdot \ell(\Gamma_R) \leq \sup_{z\in\Gamma_R} \frac{1}{|z|^4-1} \cdot \pi R = \frac{\pi R}{R^4-1},$$

which goes to 0 as $R \to \infty$.

From (13.2) we now get $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$ and we conclude that $\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

Example 13.3. Compute the integral $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{x^2-2x+2}$.

We first present the solution and then we make some comments that hopefully explain why we do as we do. Let γ_R be the interval $[-R, R]$, let Γ_R be the upper half of the circle $\{z;\, |z| = R\}$, and let $f(z) = \frac{e^{iz}}{z^2-2z+z}$. Since $z^2 - 2z + 2 = (z-a)(z-\bar{a})$, where $a = 1+i$, the function f is holomorphic in $\mathbb{C} \setminus \{a, \bar{a}\}$ and the residue at a is

$$\operatorname{Res}(f; a) = \frac{e^{iz}}{z-\bar{a}}\Big|_{z=a} = \frac{e^{ia}}{a-\bar{a}} = \frac{e^{-1+i}}{2i}$$

by Residue computation 1c. Hence, for any $R > \sqrt{2}$ (so that a is in the upper half of the disc $D(0, R)$),

$$\int_{\gamma_R+\Gamma_R} \frac{e^{iz}\, dz}{z^2-2z+2} = 2\pi i \frac{e^{-1+i}}{2i} = \pi e^{-1+i} \tag{13.3}$$

by the Residue theorem. On the other hand,

$$\int_{\gamma_R+\Gamma_R} \frac{e^{iz}\, dz}{z^2-2z+2} = \int_{-R}^{R} \frac{e^{ix}\, dx}{x^2-2x+2} + \int_{\Gamma_R} \frac{e^{iz}\, dz}{z^2-2z+2}, \tag{13.4}$$

and we claim that the second integral on the right-hand side goes to 0 as $R \to \infty$. The claim follows since

$$\begin{aligned}\left|\int_{\Gamma_R} \frac{e^{iz}\, dz}{z^2-2z+2}\right| &\leq \sup_{z\in\Gamma_R} \frac{|e^{iz}|}{|z^2-2z+2|} \cdot \ell(\Gamma_R) \leq \sup_{z\in\Gamma_R} \frac{e^{\operatorname{Re} iz}}{|z|^2-2|z|-2} \cdot \pi R \\ &\leq \sup_{z\in\Gamma_R} \frac{e^{-\operatorname{Im} z}}{R^2-2R-2} \cdot \pi R \leq \frac{\pi R}{R^2-2R-2} \to 0 \text{ as } R \to \infty,\end{aligned}$$

where the second inequality follows from Proposition 6.2 (and the Reverse triangle inequality), and the fourth inequality follows since $\operatorname{Im} z \geq 0$ on Γ_R so that $e^{-\operatorname{Im} z} \leq e^0 = 1$ for all $z \in \Gamma_R$. In view of (13.3) and (13.4) we thus have

$$\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{x^2-2x+2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}\, dx}{x^2-2x+2} = \pi e^{-1+i}. \tag{13.5}$$

However, $e^{ix} = \cos x + i\sin x$, so the integral we are interested in computing is the real part of the left-hand side of (13.5). Hence, the sought integral equals $\mathrm{Re}(\pi e^{-1+i}) = \mathrm{Re}(\pi e^{-1}(\cos 1 + i\sin 1)) = \pi e^{-1}\cos 1$.

Our comments concern the choice of f and integration contour. One might wonder why we don't choose f as $\cos z/(z^2 - 2z + z)$; this is a holomorphic function in $\mathbb{C} \setminus \{a, \bar{a}\}$ and on the real axis it is the integrand in the integral we want to compute. The step where we use the Residue theorem works just as well with this choice of f but the step where we show that the integral over Γ_R goes to 0 will not work out in a nice way. The reason is that we cannot estimate $|\cos z|$ for $z \in \Gamma_R$ in a sufficiently good way; recall that cos grows exponentially on the imaginary axis.

One might also wonder why we choose the upper half of the circle $\{|z| = R\}$ to get a closed integration contour and not the lower half. This is just a matter of taste, we can choose Γ_R as the lower part but then we need to choose f as $e^{-iz}/(z^2 - 2z + z)$ to make the step where the integral over Γ_R goes to 0 work out.

Other ways of closing up the integration contour might also work but will probably lead to more elaborate computations.

Example 13.4. Compute the integral $\displaystyle\int_0^\infty \frac{\sin^2 x\,dx}{x^2}$.

It is often a good idea first to check that the integral in question actually makes sense. In this case there is no problem; the singularity at 0 is removable (why?) and the integrand decays as $1/x^2$ as $x \to \pm\infty$ so the integral converges nicely. To compute it we begin by noticing that $\sin^2 x = (1 - \cos(2x))/2 = \mathrm{Re}(1 - e^{i2x})/2$ and set $f(z) = \dfrac{1 - e^{i2z}}{2z^2}$. However, f does not have a removable singularity at 0 since the numerator now has a zero only of order 1 at 0. Thus, f has a pole of order 1 at 0 and so we cannot choose $[-R, R]$ as part of an integration contour as we have done before, we need to take detour to avoid the origin. Let $\gamma_1(\delta, R)$ be the interval $[-R, -\delta]$, let $\gamma_2(\delta, R)$ be the interval $[\delta, R]$, let Γ_R be the upper half of the circle $\{z;\ |z| = R\}$ oriented counterclockwise, and let Γ_δ be the upper half of the circle $\{z;\ |z| = \delta\}$ oriented clockwise. Then $\gamma_1(\delta, R) + \Gamma_\delta + \gamma_2(\delta, R) + \Gamma_R$ is the boundary of the set $\Omega_R^\delta := D(0, R) \setminus \overline{D(0, \delta)}$ and f is holomorphic in an open set containing $\overline{\Omega}_R^\delta$. Therefore, by Cauchy's theorem, we have

$$\int_{\gamma_1(\delta,R)+\gamma_2(\delta,R)} f(z)\,dz + \int_{\Gamma_\delta} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz = 0. \tag{13.6}$$

The first integral should be related to the integral we want to compute and, indeed, we have

$$\begin{aligned}
\int_{\gamma_1(\delta,R)+\gamma_2(\delta,R)} f(z)\,dz &= \int_{-R}^{-\delta} \frac{1 - e^{i2x}}{2x^2}dx + \int_\delta^R \frac{1 - e^{i2x}}{2x^2}dx \\
&= \int_\delta^R \frac{1 - e^{-i2x}}{2x^2}dx + \int_\delta^R \frac{1 - e^{i2x}}{2x^2}dx \\
&= \int_\delta^R \frac{2 - (e^{i2x} + e^{-i2x})}{2x^2}dx = \int_\delta^R \frac{2 - 2\cos(2x)}{2x^2}dx \\
&= \int_\delta^R \frac{2\sin^2 x}{x^2}dx,
\end{aligned}$$

which converges to 2 times the sought integral as $\delta \to 0$ and $R \to \infty$.

The third integral in (13.6) goes to 0 as $R \to \infty$ since

$$\left|\int_{\Gamma_R} \frac{1 - e^{i2z}}{2z^2}dz\right| \le \sup_{z\in\Gamma_R} \frac{|1 - e^{i2z}|}{2|z|^2} \cdot \pi R \le \sup_{z\in\Gamma_R} \frac{1 + e^{-2\mathrm{Im}\,z}}{2R^2} \cdot \pi R \le \frac{\pi}{R}.$$

We also need to compute the second integral in (13.6). We do this similarly to the proof of Lemma 3.10. Notice that $e^{i2z} = 1 + 2iz + \mathcal{O}(|z|^2)$ by Taylor's formula. Hence,

$$\begin{aligned}
\int_{\Gamma_\delta} \frac{1-e^{i2z}}{2z^2}dz &= \int_{\Gamma_\delta} \frac{-2iz}{2z^2}dz + \int_{\Gamma_\delta} \frac{\mathcal{O}(|z|^2)}{2z^2}dz \\
&= -i\int_{\Gamma_\delta} \frac{dz}{z} + \int_{\Gamma_\delta} \mathcal{O}(1)dz = -i\int_{t=\pi}^{0} idt + \int_{\Gamma_\delta} \mathcal{O}(1)dz \\
&= -\pi + \int_{\Gamma_\delta} \mathcal{O}(1)dz.
\end{aligned}$$

The last integral here goes to 0 as $\delta \to 0$ since the integrand is bounded and the length of Γ_δ goes to 0. Thus the second integral in (13.6) goes to $-\pi$ as $\delta \to 0$. Putting all this together and letting $\delta \to 0$ and $R \to \infty$ in (13.6) we get $2\int_0^\infty \frac{\sin^2 x}{x^2}dx - \pi = 0$. The sought integral thus equals $\pi/2$.

Example 13.5. Compute the limit $\lim_{R\to\infty} \int_{-R}^{R} \frac{x-1}{x^2+1}\cos x\, dx$.

Notice that the integrand decays as $1/|x|$ as $x \to \pm\infty$ so it isn't integrable on $\mathbb{R}$ in the usual sense. Part of the exercise is to see that the given limit exists anyway. Let $f(z) = \frac{z-1}{z^2+1}e^{iz}$ and let γ_R and Γ_R be as in Examples 13.2 and 13.3. The function f is holomorphic in $\mathbb{C} \setminus \{\pm i\}$ and the residue at i can be seen, e.g., by Residue computation 2, to be $\mathrm{Res}(f; i) = e^{-1}(1+i)/2$. As in several of the examples above the Residue theorem gives

$$\int_{-R}^{R} \frac{x-1}{x^2+1}e^{ix}dx + \int_{\Gamma_R} \frac{z-1}{z^2+1}e^{iz}dz = 2\pi i \mathrm{Res}(f; i) = \pi e^{-1}(-1+i).$$

We claim that the second integral on the left-hand side goes to 0 as $R \to \infty$. Given the claim it follows that the sought limit equals $\mathrm{Re}(\pi e^{-1}(-1+i)) = -\pi e^{-1}$.

It remains to show the claim. In this case the method to estimate the integral by Proposition 2.7 and using that $|e^{iz}| \le 1$ in the upper half-plane as we have done before does not work. In fact, this method (do the computations) shows that the modulus of the second integral is bounded by $\pi R(R+1)/(R^2+1)$ which does not go to 0 as $R \to \infty$. The point is that we need to improve the estimate $|e^{iz}| \le 1$ for $z = Re^{it}$ and $0 \le t \le \pi$. First we make a preliminary computation:

$$\begin{aligned}
\left|\int_{\Gamma_R} \frac{z-1}{z^2+1}e^{iz}dz\right| &= \left|\int_0^\pi \frac{Re^{it}-1}{R^2e^{i2t}+1} iRe^{iRe^{it}}dt\right| \\
&\le \int_0^\pi \frac{R+1}{R^2-1}Re^{-R\sin t}dt = 2\frac{R^2+R}{R^2-1}\int_0^{\pi/2} e^{-R\sin t}dt.
\end{aligned}$$

To estimate the last integral we use

$$\underline{\text{Jordan's inequality:}} \qquad \text{If } 0 \le t \le \frac{\pi}{2}, \quad \text{then} \quad \frac{2}{\pi}t \le \sin t \le t.$$

In view of this we have $-\sin t \le -2t/\pi$ for $0 \le t \le \pi/2$ and so

$$\begin{aligned}
\frac{R^2+R}{R^2-1}\int_0^{\pi/2} e^{-R\sin t}dt &\le \frac{R^2+R}{R^2-1}\int_0^{\pi/2} e^{-2Rt/\pi}dt = \frac{R^2+R}{R^2-1}\left[-\pi\frac{e^{-2Rt/\pi}}{2R}\right]_0^{\pi/2} \\
&= -\frac{\pi}{2R}\frac{R^2+R}{R^2-1}(e^{-R}-1),
\end{aligned}$$

which goes to 0 as $R \to \infty$. This proves the claim.

Exercises

13.1 Compute the integral $\displaystyle\int_{\partial D(0,8)} \frac{dz}{1+e^z}$.

13.2 Compute the integral $\displaystyle\int_0^{2\pi} \frac{\sin^2 t}{5+4\cos t}dt$.

13.3 Compute the integral $\displaystyle\int_0^{\infty} \frac{\cos(2x)}{x^2+4}dx$.

13.4 Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+x+1}dx$.

13.5 Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+a^2}dx$ for all $a > 0$.

13.6 Compute the integral $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ for all $a, b \in \mathbb{R}$ such that $0 < a < b$.

13.7 Compute the integral $\displaystyle\int_0^{\infty} \frac{\sin x}{x(x^2+1)}dx$.

13.8 Compute the limit $\displaystyle\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x}dx$.

13.9 Compute the integral $\displaystyle\int_0^{\infty} \frac{dx}{1+x^n}$ for all integers $n \geq 2$.

14 Harmonic functions and the Dirichlet problem

One reason why holomorphic functions are important is that they are closely related to harmonic functions which in turn are fundamental in physics. If Ω is an open set in the plane and we place charges on its boundary, then the resulting electric potential is a harmonic function in Ω. If we instead place heat sources on the boundary and wait until the temperature distribution T becomes stationary, then T is a harmonic function in Ω. These are certainly not the only examples where harmonic functions turn up in physics.

The Dirichlet problem is a certain boundary value problem; given a continuous function on $\partial\Omega$ one seeks a continuous function on $\overline{\Omega}$ that agrees with the given one on the boundary and that is harmonic in Ω. In view of the preceding examples it is at least physically reasonable to believe that the Dirichlet problem has a unique solution.

14.1 Harmonic vs. holomorphic functions

Let $\Omega \subset \mathbb{C}$ be an open set. A C^2-smooth function u in Ω is *harmonic* in Ω if it satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in Ω. The differential operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is called the Laplace operator and is often denoted by Δ. Notice that if φ is a real-valued function, then $\Delta\varphi$ is also real-valued. It follows that the real and the imaginary parts of a harmonic function are harmonic as well. In fact, if $f = u + iv$ then $\Delta f = \Delta u + i\Delta v$ and hence, $\Delta f = 0$ if and only if $\Delta u = \Delta v = 0$.

A first indication that harmonic functions and holomorphic functions are related is given by the following result. Recalling the definition of $\partial/\partial z$ and $\partial/\partial \bar{z}$ (see Chapter 2), the proof is a straightforward computation (Exercise 14.1).

Proposition 14.1. *If φ is a C^2-smooth function then $\Delta\varphi = 4\dfrac{\partial^2\varphi}{\partial z \partial \bar{z}}$.*

It follows that holomorphic functions are harmonic. Thus, the real and the imaginary parts of a holomorphic function are harmonic. A partial converse of this statement is also true; if u is harmonic and real-valued, then there is, at least locally, a holomorphic function f such that u is the real part of f. This is the content of

Theorem 14.2. *Let $\Omega \subset \mathbb{C}$ be an open simply connected set. If u is a real-valued harmonic function in Ω then there is a real-valued harmonic function v in Ω such that $f = u + iv$ is holomorphic in Ω.*

The function v is called a *harmonic conjugate* of u. Harmonic conjugates are not unique but if v_1 and v_2 both are harmonic conjugates of u then $v_1 = v_2 + c$ for some constant c. In fact, both $u + iv_1$ and $u + iv_2$ are holomorphic so $v_1 - v_2 = -i(u + iv_1 - (u + iv_2))$ is holomorphic and real-valued. Thus $v_1 - v_2$ is constant by Proposition 3.6.

Example 14.3. If $u(x, y) = x^2 - y^2$, then $v(x, y) = 2xy$ is a harmonic conjugate of u since $u + iv = x^2 - y^2 + i2xy = (x + iy)^2 = z^2$, which is holomorphic.

Complex-valued harmonic functions also have harmonic conjugates locally, albeit complex-valued. Such harmonic conjugates are only unique up to holomorphic functions, see Exercise 14.4.

Proof of Theorem 14.2. The function $\dfrac{\partial u}{\partial z}$ is holomorphic since $\dfrac{\partial}{\partial \bar z}\dfrac{\partial u}{\partial z} = \Delta u/4 = 0$ (see Section 3.1) Therefore, since Ω is simply connected, there is a holomorphic function F in Ω such that $F' = \dfrac{\partial u}{\partial z}$ by Theorem 10.8. This means that $\dfrac{\partial(F-u)}{\partial z} = 0$. In view of Exercise 2.3 we get

$$\frac{\partial(\overline{F}-u)}{\partial \bar z} = \frac{\partial(\overline{F-u})}{\partial \bar z} = \overline{\left(\frac{\partial(F-u)}{\partial z}\right)} = 0$$

since $u = \overline{u}$. Thus $\overline{F} - u$ is holomorphic and so also $F + \overline{F} - u$ is holomorphic. But $F + \overline{F} - u$ is real-valued so it must be constant by Proposition 3.6, i.e., $F + \overline{F} - u = C$ for some real constant C. Setting $f = 2F - C$ we get a holomorphic function in Ω such that $\mathrm{Re}\, f = 2\mathrm{Re}\, F - C = F + \overline{F} - C = u$. We may thus take $v = \mathrm{Im}\, f$. $\square$

It is necessary that Ω is simply connected for Theorem 14.2 to be true. Actually, if Ω is not simply connected then there always is a harmonic function lacking harmonic conjugate in Ω but we will not prove this.

14.2 Poisson's integral formula and some consequences

Poisson's integral formula follows from Cauchy's formula and expresses a harmonic function u in unit disc in terms of the values of u on the boundary. Many properties of holomorphic functions stem from Cauchy's formula and, similarly, we will see that Poisson's integral formula implies fundamental properties of harmonic functions.

Theorem 14.4 (Poisson's integral formula)**.** *If u is continuous on $\overline{D(0,1)}$ and harmonic in $D(0,1)$, then, for any $0 \le r < 1$ and $t \in \mathbb{R}$,*

$$u(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} u(e^{i\theta})d\theta. \tag{14.1}$$

Proof. We may assume that u is real-valued. Otherwise we consider the real and imaginary parts of u separately.

Assume first that u is harmonic in an open set containing $\overline{D(0,1)}$ and let v be a harmonic conjugate of u so that $f := u + iv$ is holomorphic in an open set containing $\overline{D(0,1)}$. Fix a point $w = re^{it} \in D(0,1)$ and and set $g(z) = (1-r^2)/(1-\bar w z)$. Then g is holomorphic in an open set containing $\overline{D(0,1)}$ and $g(w) = 1$. We apply Cauchy's formula to $f(z)g(z)$:

$$\begin{aligned} f(re^{it}) &= f(w) = f(w)g(w) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)g(z)\,dz}{z-w} \\ &= \frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{1-r^2}{1-\bar w z}\frac{dz}{z-w}. \end{aligned}$$

Since $|z| = 1$ we have $z\bar z = |z|^2 = 1$ and so $z - w = (1 - w\bar z)z$. We thus get

$$\begin{aligned} f(re^{it}) &= \frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{1-r^2}{1-\bar w z}\frac{1}{1-w\bar z}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{1-r^2}{1-2\mathrm{Re}(w\bar z)+|w|^2}\frac{dz}{z}. \end{aligned}$$

We parametrize the curve $\{|z| = 1\}$ by $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then $dz/z = ie^{i\theta}d\theta/e^{i\theta} = id\theta$ and $\mathrm{Re}(w\bar z) = \mathrm{Re}(re^{it}e^{-i\theta}) = r\cos(t-\theta)$ so we get

$$f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} f(z)\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}d\theta.$$

Taking the real part we obtain (14.1).

If u is merely continuous on $\overline{D(0,1)}$ and harmonic in $D(0,1)$ we set $u_\varrho(z) := u(\varrho z)$, where $0 \le \varrho < 1$. Then u_ϱ is harmonic in an open set containing $\overline{D(0,1)}$ and so (14.1) holds with u replaced by u_ϱ. But u is uniformly continuous on $\overline{D(0,1)}$ so for any given $\epsilon > 0$ there is a $\delta > 0$ such that $|u(z) - u_\varrho(z)| = |u(z) - u(\varrho z)| \le \epsilon$ if $|z - \varrho z| \le \delta$; then (14.1) follows by letting $\varrho \to 1$. □

Corollary 14.5 (Mean value property of harmonic functions)**.** *If u is harmonic in an open set containing $\overline{D(0,R)}$ then $u(0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(\varrho e^{i\theta})d\theta$ if $0 \le \varrho \le R$.*

Proof. Let $u_\varrho(z) := u(\varrho z)$, where $0 \le \varrho \le R$. Since u_ϱ is harmonic in an open set containing $\overline{D(0,R)}$ Poisson's integral formula gives

$$u(0) = u_\varrho(0) = \frac{1}{2\pi}\int_0^{2\pi} u_\varrho(e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(\varrho e^{i\theta})\,d\theta.$$

□

Theorem 14.6 (Maximum principle for harmonic functions)**.** *Let $\Omega \subset \mathbb{C}$ be an open bounded connected set. If u is a continuous function on $\overline{\Omega}$ that is harmonic in Ω, then $|u|$ attains its maximum on the boundary, i.e., $\sup_{z\in\overline{\Omega}} |u(z)| = \sup_{z\in\partial\Omega} |u(z)|$.*

Proof. The proof relies on the Mean value property and is essentially the same as the proof of the Maximum principle in Chapter 11; we recall the main points.

Let $M = \sup_{z\in\overline{\Omega}} |u(z)|$. Since Ω is bounded and $|u|$ is continuous on $\overline{\Omega}$, M is finite and there is an $a \in \overline{\Omega}$ such that $M = |u(a)|$. If $a \in \partial\Omega$ we are done so assume that $a \in \Omega$. Let $A = \{z \in \Omega;\ |u(z)| = M\}$; since $a \in A$, A is non-empty. Moreover, since $|u|$ is continuous it follows that A is closed (why?). If A in addition is open it follows that $A = \Omega$ since Ω is connected (why?). But then $|u|$ is a constant function so it attains its maximum (=only value) on the boundary.

It remains to show that A is open. Let $b \in A$ and choose $R > 0$ such that $\overline{D(0,R)} \subset \Omega$. Using the Mean value property of harmonic functions and computing as in (11.1) (with f replaced by u, a replaced by b, and r replaced by ϱ) we get

$$\frac{1}{2\pi}\int_0^{2\pi} |u(b)| - |u(b + \varrho e^{i\theta})|\,d\theta = 0$$

for all $\varrho \le R$. But since b is a maximum point, the integrand must be non-negative, and a non-negative function cannot have integral 0 unless the function is 0. Hence, $|u(b)| - |u(b+\varrho e^{i\theta})| = 0$ for all $\theta \in [0, 2\pi]$ and all $\varrho \le R$, i.e., $|u|$ is constant in $\overline{D(b,R)}$. Thus, $D(b,R) \subset A$, which shows that A is open. □

14.3 The Dirichlet problem and the Riemann mapping theorem revisited

By the Dirichlet problem we will mean the following: Let $\Omega \subset \mathbb{C}$ be an open bounded and connected set and let v be a continuous function on $\partial\Omega$. Find a continuous function u on $\overline{\Omega}$ that is harmonic in Ω and agrees with v on $\partial\Omega$.

If the Dirichlet problem is solvable then the solution is unique; this follows from the Maximum principle. In fact, if u_1 and u_2 are continuous on $\overline{\Omega}$, harmonic in Ω, and $u_1 = u_2$ on $\partial\Omega$, then $u_1 - u_2$ is continuous on $\overline{\Omega}$, harmonic in Ω, and 0 on $\partial\Omega$. Hence, $|u_1 - u_2| = 0$ on $\partial\Omega$ and so, by the Maximum principle, $|u_1 - u_2| \le 0$ in Ω and we thus get $u_1 = u_2$ in Ω.

For $\Omega = D(0,1)$ the Dirichlet problem is solvable. Let υ be a continuous function on $\partial\Omega$. Notice that if u is a solution of the corresponding Dirichlet problem then, by Poisson's formula, the values of u in $D(0,1)$ are given by

$$u(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\upsilon(e^{i\theta})d\theta. \tag{14.2}$$

The integral on the right-hand side only involves the given function υ so if we want to find a solution of the Dirichlet problem it is natural to define a tentative solution as the integral on the right-hand side. This will at least give us a harmonic function u in $D(0,1)$. In fact, if $z = re^{it}$, then (Exercise 14.2)

$$\frac{1-r^2}{1-2r\cos(t-\theta)+r^2} = \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right), \tag{14.3}$$

so the integrand in (14.2), as a function of $re^{it} \in D(0,1)$, is a harmonic function. It is also true that u defined by (14.2) in $D(0,1)$ has a continuous extension to $\overline{D(0,1)}$ that agrees with υ on $\partial D(0,1)$, but we will not show this.

Let now Ω be an open bounded and simply connected set. By the Riemann mapping theorem there is holomorphic bijective map $\varphi\colon \Omega \to D(0,1)$. If $\partial\Omega$ is sufficiently nice it is not unreasonable to believe that φ can be extended to a continuous bijective map $\overline{\Omega} \to \overline{D(0,1)}$. In this case we can solve the Dirichlet problem in Ω as follows. Let ν be a continuous function on $\partial\Omega$. Then $\upsilon := \nu \circ \varphi^{-1}_{\restriction \partial D(0,1)}$ is a continuous function on $\partial D(0,1)$. Since we can solve the Dirichlet problem in $D(0,1)$ there is a continuous function u on $\overline{D(0,1)}$ that is harmonic in $D(0,1)$ and agrees with υ on $\partial D(0,1)$. Then $v := u \circ \varphi$ is a continuous function on $\overline{\Omega}$ that agrees with ν on $\partial\Omega$ (why?) and it is harmonic in Ω since $\Delta(u \circ \varphi) = |\varphi'|^2 \Delta u = 0$, see Exercise 14.3.

The Riemann mapping theorem thus indicates that the Dirichlet problem is solvable in sufficiently nice simply connected open sets. On the other hand, if the Dirichlet problem is solvable in simply connected open set, we can almost prove the Riemann mapping theorem. Since the Dirichlet problem should be solvable for physical reasons, see the introduction to this chapter, we at least get a physical justification of the Riemann mapping theorem.

We end these notes by a "hand-waving" proof of the Riemann mapping theorem given that the Dirichlet problem can be solved. Let Ω be an open bounded and simply connected set and pick a point $a \in \Omega$. Then $-\log|z-a|$ defines a continuous function on $\partial\Omega$. If the Dirichlet problem can be solved in Ω we find a continuous function u on $\overline{\Omega}$ that agrees with $-\log|z-a|$ on $\partial\Omega$ and is harmonic in Ω. Since Ω is simply connected there is a harmonic conjugate v of u by Theorem 14.2. Then $\varphi(z) := (z-a)e^{u(z)+iv(z)}$ is holomorphic in Ω and has precisely 1 zero, multiplicity counted, since $e^{u(z)+iv(z)} \neq 0$. For $z \in \partial\Omega$ we have

$$|\varphi(z)| = |z-a|e^{\mathrm{Re}\, f(z)} = |z-a|e^{u(z)} = |z-a|e^{-\log|z-a|} = 1$$

and it follows from the Maximum principle that φ maps Ω to $D(0,1)$. We will show that φ is bijective under the additional assumption that φ can be extended to a holomorphic function in some open set containing $\overline{\Omega}$ and that $\partial\Omega$ is piecewise smooth. In this case φ maps $\partial\Omega$ to $\partial D(0,1)$ and it follows that

$$\psi(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\varphi'(z)\,dz}{\varphi(z)-w}$$

is a continuous function in $D(0,1)$. Moreover, by the Argument principle, $\psi(w)$ is the number of points $z \in \Omega$ such that $\varphi(z) = w$. We want to show that $\psi(w) = 1$ for all $w \in D(0,1)$. But ψ is continuous and integer-valued so it must be constant, and this constant has to be 1 since $\psi(0) = 1$.

Exercises

14.1 Show Proposition 14.1.

14.2 Show (14.3), where $z = re^{it}$.

14.3 Let $u(w)$ be harmonic and let $\varphi(z)$ be holomorphic. Show, using the Chain rule, that $\dfrac{\partial^2 (u \circ \varphi)}{\partial z \partial \bar{z}} = |\varphi'|^2 \dfrac{\partial^2 u}{\partial w \partial \bar{w}}$.

14.4 Let f be a possibly complex-valued harmonic function in a simply connected open set Ω. Show that there is a harmonic function g in Ω such that $f + ig$ is holomorphic in Ω. Show also that if g_1 and g_2 are harmonic functions such that $f + ig_1$ and $f + ig_2$ are holomorphic, then $g_1 - g_2$ is holomorphic.

14.5 Let Ω be a bounded open set with piecewise smooth boundary and assume that the Dirichlet problem is solvable in Ω. Set $\nu_a(z) = 1/(z-a)$ for $z \in \partial\Omega$ and $a \in \Omega$ and let $v_a(z)$ be the solution of the corresponding Dirichlet problem. Show that $K_a(z) := \dfrac{1}{\pi} \dfrac{\partial \overline{v}}{\partial z}$ is holomorphic in Ω. Show also that if g is holomorphic in an open set containing $\overline{\Omega}$ then

$$g(a) = \int_\Omega g(z) \overline{K_a(z)}\, dxdy.$$

(Hint: Cauchy's formula and (2.9).)

Appendix A

Complex numbers

A complex number z is an ordered pair (x, y) of real numbers $x, y \in \mathbb{R}$. We usually write $z = x + iy$, where the i in front of y here means that y is the second element of the ordered pair. We identify the set $\mathbb{C}$ of complex numbers with $\mathbb{R}^2$ and we picture the complex number $z = x+iy$ as the point (x, y) in $\mathbb{R}^2$. The x-axis is called the real axis and the y-axis is called the imaginary axis. The standard projections on these axes are denoted Re and Im respectively, so that $\operatorname{Re}(x + iy) = x$ and $\operatorname{Im}(x + iy) = y$. For a complex number z, $\operatorname{Re} z$ is called the real part of z and $\operatorname{Im} z$ is called the imaginary part of z. (The imaginary part of a complex number is thus a real number!) The set of real numbers is then naturally identified with the real axis by identifying $x \in \mathbb{R}$ with $(x, 0)$. The conjugate, $\bar{z}$, of a complex number $z = x + iy$ is $\bar{z} := x - iy$. The absolute value, or modulus, of $z = x + iy$ is defined to be $|z| := \sqrt{x^2 + y^2}$ and is thus the Euclidean distance between 0 and (x, y) in $\mathbb{R}^2$.

A complex number can also be represented using polar coordinates on $\mathbb{R}^2$. If $z = x + iy$ is a complex number, let $r = |z|$ and let θ be the angle between the positive real axis and the ray from 0 through (x, y). Then $(x, y) = (r\cos\theta, r\sin\theta)$, and the polar representation of z is

$$z = r\cos\theta + ir\sin\theta = re^{i\theta}. \tag{14.4}$$

At this point $e^{i\theta}$ is just a notation but in Chapter 6 we will define the exponential of any complex number (in a natural way extending the exponential of real numbers) and show that (14.4) holds.

Arithmetics of complex numbers

Addition and multiplication of two complex numbers $z = x + iy$ and $w = u + iv$ are defined as follows:

$$\begin{aligned} z + w &= (x + u) + i(y + v), \\ z \cdot w &= (xu - yv) + i(xv + yu). \end{aligned}$$

Addition is easy to visualize, thinking of z and w as vectors in $\mathbb{R}^2$ starting at 0 and ending at (x, y) and (u, v) respectively, the sum $z + w$ corresponds to the vector sum of these two vectors. To visualize multiplication, it is convenient to use polar representation. Let $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$. Using the definition of complex multiplication and the addition formulas for sin and cos we get

$$\begin{aligned} z \cdot w &= (r\cos\theta + ir\sin\theta = re^{i\theta}) \cdot (\rho\cos\varphi + i\rho\sin\varphi) \qquad (14.5) \\ &= r\rho\left((\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi)\right) \\ &= r\rho\left(\cos(\theta+\varphi) + i\sin(\theta+\varphi)\right). \end{aligned}$$

Thus, the modulus of the product $z \cdot w$ is the product of the moduli $|z|$ and $|w|$, and the angle corresponding to the product $z \cdot w$ is the sum of the angles corresponding to z and w. In particular, if $|w| = 1$, then you get $z \cdot w$ by rotating (the vector corresponding to) z counterclockwise by the angle corresponding to w.

The computation in (14.5) and induction (do the details!) shows that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for any positive integer n. This equality is called de Moivre's formula. In our new notation it can alternatively be written $(e^{i\theta})^n = e^{in\theta}$, which give some justification for the notation.

The first eleven of the following computation rules are easily checked (do it!) from the definitions of complex addition and multiplication.

(1) $z + w = w + z$

(2) $(z + w) + \zeta = z + (w + \zeta)$

(3) $z \cdot w = w \cdot z$

(4) $(z \cdot w) \cdot \zeta = z \cdot (w \cdot \zeta)$

(5) $z \cdot (w + \zeta) = z \cdot w + z \cdot \zeta$

(6) $\bar{\bar{z}} = z$

(7) $z + \bar{z} = 2\mathrm{Re}\, z$, $z - \bar{z} = 2i\mathrm{Im}\, z$

(8) $\overline{z + w} = \bar{z} + \bar{w}$

(9) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

(10) $|\bar{z}| = |z|$

(11) $z \cdot \bar{z} = |z|^2$

(12) $|\mathrm{Re}\, z| \leq |z|$, $|\mathrm{Im}\, z| \leq |z|$

(13) $|z + w| \leq |z| + |w|$

(14) $|z + w| \geq \big||z| - |w|\big|$.

To show (12), recall that $|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$ and notice that the terms are non-negative. Thus, $(\mathrm{Re}\, z)^2 = |z|^2 - (\mathrm{Im}\, z)^2 \leq |z|^2$ and so $|\mathrm{Re}\, z| \leq |z|$. We get $|\mathrm{Im}\, z| \leq |z|$ in a similar way.

To show (13), which is known as the Triangle inequality, we use (some of) the previous computation rules:

$$
\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\bar{z} + \bar{w}) \\
&= |z|^2 + z\bar{w} + \bar{z}w + |w|^2 = |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \\
&= |z|^2 + 2\mathrm{Re}\,(z\bar{w}) + |w|^2 \leq |z|^2 + 2|z| \cdot |w| + |w|^2 \\
&= (|z|^2 + |w|^2)^2.
\end{aligned}
$$

The Reverse triangle inequality, (14), follows from the Triangle inequality in the same way as in real calculus.

Please note that inequalities can only occur between real numbers, it has no meaning to write "$z \leq w$" if any of z and w is complex!

Division of complex numbers is defined as follows. First, if $z = x + iy$ and $t \in \mathbb{R} \setminus \{0\}$, then we define $z/t = (1/t) \cdot z = (x/t) + i(y/t)$. Second, if $z \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$, then z/w is defined to be the unique complex number q satisfying $w \cdot q = z$. To compute q we multiply by $\bar{w}$ and get $|w|^2 \cdot q = z \cdot \bar{w}$; dividing by the non-zero real number $|w|^2$ we obtain $q = z \cdot \bar{w}/|w|^2$. In particular, $w^{-1} := 1/w = \bar{w}/|w|^2$; it is straightforward to check that de Moivre's formula holds also for negative integers.

Polynomials

A polynomial of the complex variable z is a function $\mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = \sum_{k=0}^{n} a_k z^k,$$

where the coefficients a_k are complex numbers. The integer n is called the degree of the polynomial. Notice that, setting $z = x + iy$, p becomes a polynomial of the two real variables x and y (with complex coefficients). However, not every polynomial of two real variables can be obtained in this way, cf. the Informal introduction above.

The solutions of a polynomial equation $p(z) = 0$ are called the roots of the polynomials. The Fundamental theorem of algebra, which is proved (by analytic techniques!) in Chapter 5, states that every polynomial has a root in $\mathbb{C}$. (By the standard algorithm of polynomial division it then follows that a polynomial of degree n has precisely n roots in $\mathbb{C}$ taking multiplicity into account, cf. Chapter ??? above.)

Example 14.7 (Roots of unity)**.** Find the roots of the polynomial $z^n - 1$, $n \geq 0$; these are called the n^{th} roots of unity.
Solution: We use polar representation and write $z = re^{i\theta}$. By de Moivre's formula we are looking for $r \geq 0$ and $\theta \in \mathbb{R}$ such that $r^n e^{in\theta} = 1$. From the identity $\sin^2\theta + \cos^2\theta = 1$ and the computation rule (11) it follows that $|e^{in\theta}| = 1$. Thus, $1 = |1| = |r^n e^{in\theta}| = r^n$ so $r = 1$; it remains to find $\theta \in \mathbb{R}$ such that $e^{in\theta} = 1$, i.e., such that $\cos(n\theta) + i\sin(n\theta) = 1$. Taking the real and imaginary parts of this equation we get

$$\begin{cases} \cos(n\theta) = 1 \\ \sin(n\theta) = 0. \end{cases}$$

From calculus we know that the solutions of these equations are $n\theta = 2\pi k$, $k \in \mathbb{Z}$, so $\theta = 2\pi k/n$, $k \in \mathbb{Z}$. For $k = 0, 1, \dots, n-1$, $e^{i2\pi k/n}$ are different complex numbers of modulus 1 lying on the vertices of the regular n-gon centered at 0 with one vertex at $(1, 0)$. Now, any $k \in \mathbb{Z}$ can be written $k = k_0 + \ell \cdot n$ for some integer ℓ and some $k_0 \in \{0, 1, \dots, n-1\}$ and so

$$e^{i2\pi k/n} = e^{i2\pi(k_0+\ell\cdot n)/n} = e^{i2\pi k_0/n + i2\pi\ell} = e^{i2\pi k_0/n} \cdot e^{i2\pi\ell} = e^{i2\pi k_0/n},$$

where the third equality follows from the addition formulas for sin and cos, cf. (14.5), and the forth since $e^{i2\pi\ell} = \cos(2\pi\ell) + i\sin(2\pi\ell) = 1$. Hence we do not get any new solutions.

In conclusion, the roots of $z^n - 1$ are the n complex numbers $1, e^{i2\pi/n}, e^{i2\pi 2/n}, \dots, e^{i2\pi(n-1)/n}$.

Appendix B

Open and closed subsets of $\mathbb{C}$ and of the Riemann sphere

A subset $S \subset \mathbb{C}$ is *open* in $\mathbb{C}$ if for each $a \in S$ there some $r > 0$ such that the disc centered at a with radius r is contained in S, i.e., $\{z \in \mathbb{C};\ |z-a| < r\} \subset S$.

Example 14.8. Let $S = \{z \in \mathbb{C};\ |z| < 1\}$. Then S is open. In fact, if $a \in S$, then $|a| < 1$ and so there is some $r > 0$ such that $|a| + r < 1$. Hence, if z satisfies $|z-a| < r$, then it follows from the Triangle inequality that $|z| = |z-a+a| \leq |z-a| + |a| < r + |a| < 1$, i.e., the disc centered at a with radius r is contained in S.

It is straightforward to check from the definition that the following holds.

- The empty set, $\emptyset$, and $\mathbb{C}$ are open,
- if $S_1, \ldots, S_n$ are open subsets of $\mathbb{C}$ then $S_1 \cap \cdots \cap S_n$ is open,
- if $\{S_i\}_{i \in I}$ is an arbitrary family of open subsets then $\cup_{i \in I} S_i$ is open.

Notice also that any open subset of $\mathbb{C}$ is a union of discs since if S is open, then for each $a \in S$ there is an $r_a > 0$ such that $D(a, r_a) := \{z \in \mathbb{C};\ |z-a| < r_a\} \subset S$. It follows that, $S = \cup_{a \in S} D(a, r_a)$.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane, or the Riemann sphere. A subset $S \subset \widehat{\mathbb{C}}$ is open in $\widehat{\mathbb{C}}$ if S is a union of discs $D(a, r) \subset \mathbb{C}$ and subsets of the form $\{\infty\} \cup \{z \in \mathbb{C};\ |z| > r\}$. If we identify $\widehat{\mathbb{C}}$ with the unit sphere in $\mathbb{R}^3$ as in Chapter 1, then it is not hard to believe (and it is indeed true) that the open subsets of $\widehat{\mathbb{C}}$ precisely correspond to the intersection between the unit sphere and the open subsets of $\mathbb{R}^3$.

A subset S of $\mathbb{C}$ (respectively $\widehat{\mathbb{C}}$) is *closed* in $\mathbb{C}$ (resp. $\widehat{\mathbb{C}}$) if $\mathbb{C} \setminus S$ (resp. $\widehat{\mathbb{C}} \setminus S$) is open.

Let S be a subset of $\mathbb{C}$ or $\widehat{\mathbb{C}}$. A point $a \in \mathbb{C}$ is a *limit point* of S if $S \cap \{z \in \mathbb{C};\ 0 < |z-a| < r\}$ is non-empty for all $r > 0$. The point ∞ is a limit point of S if $S \cap \{z \in \mathbb{C};\ |z| > r\}$ is non-empty for all $r > 0$.

The *closure*, $\overline{S}$, of a set S is the union of S and all its limit points.

Proposition 14.9. *Let $S \subset \mathbb{C}$. Then (1) $a \in \overline{S}$ if and only if $S \cap U$ is non-empty for every open set U containing a, (2) $\overline{S}$ is closed, and (3) S is closed if and only if $S = \overline{S}$.*

The *boundary* of an open subset $S \subset \mathbb{C}$ is defined, and denoted, by $\partial S := \overline{S} \setminus S$.

Some topology and analysis in $\mathbb{C}$ and the Riemann sphere

Specifying a topology on some given set X amounts to defining what subsets of X are to be called open, and this family of subsets should have the three properties listed after Example 14.8 above, with $\mathbb{C}$ in the first property replaced by X. Having a topology on X is the minimum requirement to be able to speak about convergence, continuity, and basic geometric features of X. The families of open subsets of $\mathbb{C}$ and $\widehat{\mathbb{C}}$ defined above are topologies on $\mathbb{C}$ resp. $\widehat{\mathbb{C}}$.

Let $S \subset \mathbb{C}$; S is said to be *compact* if it is closed and *bounded*, which means that there is some $r > 0$ such that $S \subset \{z \in \mathbb{C};\ |z| < r\}$. S is *convex* if for any two points $a, b \in S$, the line segment connection a and b is contained in S. If S is open then S is said to be *connected* if S cannot be written as the union of two disjoint non-empty open sets $S_1, S_2 \subset S$; S is said to be *path-connected* if for any two points $a, b \in S$ there is a continuous map $\gamma\colon [0,1] \to \mathbb{C}$ such that $\gamma(0) = a$, $\gamma(1) = b$, and $\gamma([0,1]) \subset S$. It can be shown that an open subset of $\mathbb{C}$ is connected if and only if it is path-connected.

A *sequence* of complex numbers (points on $\widehat{\mathbb{C}}$) is an enumeration, with repetitions allowed, of a set of complex numbers (points on $\widehat{\mathbb{C}}$); we write, e.g., $\{z_1, z_2, \ldots\}$, $\{z_n\}_{n=1}^{\infty}$, or simply $\{z_n\}$. We say that a sequence $\{z_n\}$ *converges* to $a \in \mathbb{C}$ if for every $\epsilon > 0$ there is some N_ϵ such that if $n \geq N$ then $|z_n - a| < \epsilon$. This is the same thing as saying that for every open set U containing a, z_n is in U if n is large enough. Phrased in this way it also makes sense if $a = \infty \in \widehat{\mathbb{C}}$. We write $\lim_{n\to\infty} z_n = a$ to denote that the sequence $\{z_n\}$ converges to a.

Let f be a function defined on an open subset S of $\mathbb{C}$ or $\widehat{\mathbb{C}}$ with values in either $\mathbb{C}$ or $\widehat{\mathbb{C}}$. Then f is *continuous at a point* $a \in S$ if for any sequence $\{z_n\}$ of points in S converging to a the sequence $\{f(z_n)\}$ converges to $f(a)$. This is the same as saying that for any open set U containing $f(a)$ there is an open set $V \subset S$ containing a such that $f(V) \subset U$. If f is continuous at every point of S then we say that f is *continuous* (on S). In general, we say that $f(z)$ has a limit as $z \to a$, or that $\lim_{z\to a} f(z)$ exists, if $\{f(z_n)\}$ converges to the same fixed point for any sequence $\{z_n\}$ (inside S) converging to a; $\lim_{z\to a} f(z)$ then denotes this point. Notice that in this terminology, f is continuous at a if and only if $\lim_{z\to a} f(z)$ exists; necessarily then $\lim_{z\to a} f(z) = f(a)$ since the "constant" sequence $z_n = a$ converges to a.

The following lemma says in particular that convergence of sequences of complex numbers is the same as convergence of the corresponding sequences of points in $\mathbb{R}^2$, and that a function $f\colon \mathbb{C} \to \mathbb{C}$ is continuous if and only if it is continuous considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$.

Lemma 14.10. (a) *A sequence $\{z_n\}$ of complex numbers converges if and only if the sequences $\{\mathrm{Re}\, z_n\}$ and $\{\mathrm{Im}\, z_n\}$ of real numbers converge. Moreover, if $\{z_n\}$ converges to a, then $\{\mathrm{Re}\, z_n\}$ converges to $\mathrm{Re}\, a$, $\{\mathrm{Im}\, z_n\}$ converges to $\mathrm{Im}\, a$, and $\{|z_n|\}$ converges to $|a|$.*

(b) *Let $f(z) = u(z) + iv(z)$, where $u(z) = \mathrm{Re}\, f(z)$ and $v(z) = \mathrm{Im}\, f(z)$, be a map from an open set $S \subset \mathbb{C}$ to $\mathbb{C}$. Then $\lim_{z\to a} f(z)$ exists if and only if both $\lim_{z\to a} u(z)$ and $\lim_{z\to a} v(z)$ exist. Moreover, if $\lim_{z\to a} f(z) = A$, then $\lim_{z\to a} u(z) = \mathrm{Re}\, A$, $\lim_{z\to a} v(z) = \mathrm{Im}\, A$, $\lim_{z\to a} |f(z)| = |A|$, and $\lim_{z\to a} \overline{f(z)} = \bar{A}$.*

(c) *Let $f(z) = u(z) + iv(z)$ be as in (b). Then f is continuous at $a \in S$ if and only if both u and v are continuous at a. Moreover, if f is continuous then $|f|$ and $\bar{f}$ are continuous.*

Proof. The proofs of (a), (b), and (c) are similar; we prove (c) and leave (a) and (b) as exercises.

Assume that f is continuous at a and let $\epsilon > 0$ be given. From the definition of continuity above it follows (how?) that there is a $\delta > 0$ such that if $|z - a| < \delta$ then $|f(z) - f(a)| < \epsilon$. Hence, by computation rule (12) in Appendix A, if $|z - a| < \delta$ we have

$$|u(z) - u(a)| = |\mathrm{Re}(f(z) - f(a))| \leq |f(z) - f(a)| < \epsilon.$$

Similarly, $|v(z) - v(a)| < \epsilon$. Thus u and v are continuous at a.

Conversely, assume that u and v are continuous and let $\epsilon > 0$ be given. By assumption it then follows that there is a $\delta > 0$ such that if $|z - a| < \delta$ then $|u(z) - u(a)| < \epsilon/2$ and $|v(z) - v(a)| < \epsilon/2$. Thus, if $|z - a| < \delta$ we get, using the Triangle inequality, that

$$\begin{aligned} |f(z) - f(a)| &= |u(z) - u(a) + i(v(z) - v(a))| \\ &\leq |u(z) - u(a)| + |v(z) - v(a)| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned} \tag{14.6}$$

Hence, f is continuous at a.

For the last statement of (c), assume again that f is continuous at a. We then know that u and v are continuous and it follows in the usual way that $u^2 + v^2$ is continuous, and non-negative. Since the square-root function is continuous $[0, \infty) \to \mathbb{R}$ it follows in the standard way that that the composition $\sqrt{u^2 + v^2} = |f|$ is continuous. Finally, $\bar{f} = u - iv$ and computing as in (14.6) shows that $\bar{f}$ is continuous at a. $\square$