Suggested solutions

Analytic function theory, MMG700, Exam Aug 24 2018

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1. See the literature.
2. See the literature.
3. (a) See the literature.

   (b) The zeros of $\sin z$ in $D(0,4)$ are $z = 0$ and $z = \pm\pi$ and these are zeros of order 1. Thus, the singularities of $z/\sin z$ in $D(0,4)$ are at $z = 0$, which is removable, and $z = \pm\pi$, which are poles of order 1. By, e.g., residue rule 2 we have $\mathrm{Res}(z/\sin z, \pm\pi) = \pm\pi/\cos(\pm\pi) = \mp\pi$ and so Cauchy's residue theorem gives

$$\int_{|z|=4} \frac{z\,dz}{\sin z} = 2\pi i(\mathrm{Res}(z/\sin z, \pi) + \mathrm{Res}(z/\sin z, -\pi)) = 2\pi i(-\pi + \pi) = 0.$$

4. Set $f(z) = 4z^4 + 1$ and $g(z) = e^z$. For z with $|z| = 1$ we have

$$|f(z)| = |4z^4 + 1| \geq 4|z|^4 - 1 = 3$$

   and

$$|g(z)| = |e^z| = e^{\mathrm{Re}\, z} \leq e^1 = e.$$

   Since $e < 3$ we see that $|g(z)| < |f(z)|$ if $|z| = 1$. By Rouche's theorem f and $f + g$ have the same number of zeros in $D(0,1)$. Since $f(z) = 4z^4 + 1$ has 4 zeros in $D(0,1)$, so does $4z^4 + 1 + e^z$.

5. The solutions of $z^2 - i\sqrt{2}z - 1 = 0$ are $a := (1+i)/\sqrt{2} = e^{i\pi/4}$ and $b := (-1 + i)/\sqrt{2} = e^{i3\pi/4}$ so $z^2 - i\sqrt{2}z - 1 = (z-a)(z-b)$. By a partial fraction decomposition we get

$$\frac{1}{z^2 - i\sqrt{2}z - 1} = \frac{1}{\sqrt{2}}\Big(\frac{1}{z-a} - \frac{1}{z-b}\Big).$$

   Since $|a| = |b| = 1$ we have $|a/z| < 1$ and $|b/z| < 1$ if $|z| > 1$. Hence, by the formula for a geometric series

$$\frac{1}{z-a} = \frac{1}{z}\frac{1}{1-(a/z)} = \frac{1}{z}\sum_{k=0}^{\infty}\Big(\frac{a}{z}\Big)^k$$

   if $|z| > 1$ and similarly for $1/(z-b)$. If $|z| > 1$ we thus have

$$\begin{aligned}\frac{1}{z^2 - i\sqrt{2}z - 1} &= \frac{1}{\sqrt{2}}\Big(\frac{1}{z-a} - \frac{1}{z-b}\Big) = \frac{1}{\sqrt{2}}\Big(\sum_{k=0}^{\infty}\frac{a^k}{z^{k+1}} - \sum_{k=0}^{\infty}\frac{b^k}{z^{k+1}}\Big)\\ &= \frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{e^{ik\pi/4} - e^{i3k\pi/4}}{z^{k+1}},\end{aligned}$$

   which is the desired Laurent expansion.

6. Let $\Omega_1 = \{z;\, |z-i| < 1, \operatorname{Re} z > 0\}$ and set $f_1(z) = 1/z$. Since f_1 is a Möbius transformation and

$$f_1(0) = \infty, \quad f_1(i) = -i, \quad f_1(2i) = -i/2, \quad f_1(1+i) = (1-i)/2$$

it follows that f_1 maps the boundary of Ω_1 (the line segment $\{x = 0,\, 0 \le y \le 2\}$ together with the half circle $\{|z-i| = 1, \operatorname{Re} z \ge 0\}$) to the two rays $\{x = 0,\, y \le -1/2\}$ and $\{x > 0,\, y = -1/2\}$ and that $f_1(\Omega_1) = \{x > 0,\, y < -1/2\} =: \Omega_2$.

Let $f_2(z) = z + i/2$. Then $f_2(\Omega_2)$ is the fourth quadrant, that we denote Ω_3.

Let $f_3(z) = -z^2$. Then f_3 is a conformal map of Ω_3 onto the upper half-plane.

Thus, the composition

$$f(z) := f_3 \circ f_2 \circ f_1(z) = -(1/z + i/2)^2 = \cdots = \frac{z^2 - 4iz - 4}{4z^2}$$

is a conformal map of Ω_1 onto the upper half-plane.

7. Set $F(t) = \displaystyle\int_{-\infty}^{\infty} \frac{e^{-itx}\,dx}{x^2+1}$; since $|e^{-itx}| = 1$ for any $t, x \in \mathbb{R}$ the integral converges (it is the Fourier transform of $1/(x^2+1)$). Moreover, by the change of variables $x \mapsto -u$ in the integral one sees that $F(-t) = F(t)$, that is, F is an even function.

Assume that $t \ge 0$ and set $g(z) = e^{-itz}/(z^2+1)$. $g(z)$ is holomorphic in $\mathbb{C}$ except for first order poles at $z = \pm i$. The residues at these points are

$$\operatorname{Res}(g; \pm i) = \left.\frac{e^{-itz}}{2z}\right|_{z=\pm i} = \mp\frac{ie^{\pm t}}{2}$$

by, e.g., residue rule 2. Let $\gamma_1(R)$ be the interval $\{-R \le x \le R,\, y = 0\}$ oriented from left to right and let $\gamma_2(R)$ be the lower half of the circle $\{|z| = R\}$ oriented clockwise. The region Ω_R bounded by $\gamma_1(R)$ and $\gamma_2(R)$ contains only the pole at $z = -i$ so Cauchy's residue theorem gives

$$\int_{\gamma_1(R)+\gamma_2(R)} g(z)\,dz = -2\pi i(ie^{-t}/2) = \pi e^{-t}; \qquad (*)$$

notice the extra minus-sign coming from that $\gamma_1(R) + \gamma_2(R) = -\partial\Omega_R$.

On the other hand,

$$\int_{\gamma_1(R)+\gamma_2(R)} g(z)\,dz = \int_{-R}^{R} \frac{e^{-itx}\,dx}{x^2+1} + \int_{\gamma_2(R)} \frac{e^{-itz}\,dz}{z^2+1}. \qquad (**)$$

But if $z \in \gamma_2(R)$ then $t\operatorname{Im} z \le 0$ since $t \ge 0$ and so

$$\left|\frac{e^{-itz}\,dz}{z^2+1}\right| = \frac{e^{\operatorname{Re}(-itz)}}{|z^2+1|} = \frac{e^{t\operatorname{Im} z}}{|z^2+1|} \le \frac{1}{|z^2+1|} \le \frac{1}{R^2-1}.$$

Hence,

$$\left|\int_{\gamma_2(R)} \frac{e^{-itz}\,dz}{z^2+1}\right| \le \frac{\operatorname{length}(\gamma_2(R))}{R^2-1} \to 0,\ R \to \infty,$$

that is, the second integral on the right-hand side of $(**)$ goes to 0 as $R \to \infty$. In view of $(*)$ and $(**)$ we get

$$\int_{-\infty}^{\infty} \frac{e^{-itx}\,dx}{x^2+1} = \pi e^{-t}, \quad t \geq 0.$$

Since the left-hand side, called $F(t)$ above, is an even function it follows that $F(t) = \pi e^{-|t|}$ for $t \in \mathbb{R}$.

8. Since $\operatorname{Re} f > 1$ we have $f \neq 0$ so $1/f$ is holomorphic where f is, that is, in $D(0,1) \setminus \{0\}$. We need to show that $1/f$ has a removable singularity at $z = 0$.

   The assumption that $\operatorname{Re} f > 1$ means that the image of f is contained in the set $\Omega := \{w;\, \operatorname{Re} w > 1\}$. The Möbius transformation $w \mapsto 1/w$ maps the line $\operatorname{Re} w = 1$ to the circle through 0, 1, and $(1-i)/2$ and Ω is mapped to the disc Δ bounded by this circle. Hence, $1/f(z)$ maps $D(0,1) \setminus \{0\}$ into Δ. In particular, $1/f(z)$ is a bounded function on $D(0,1) \setminus \{0\}$. By Proposition 12.3 in the notes $1/f(z)$ has a removable singularity at $z = 0$.