# Suggested solutions

Analytic function theory, MMG700, Exam Jan 15 2019

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1. See the literature.
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3. (a) By, e.g., rule 1c we have

$$\operatorname{Res}\left(\frac{\sin z}{z-\pi};\pi\right) = \lim_{z\to\pi} \sin z = \sin\pi = 0.$$

   (b) Since $\sin z$ has a zero of multiplicity 1 at 0 we get by rule 2 that

$$\operatorname{Res}\left(\frac{e^z}{\sin z};0\right) = \frac{e^z}{\cos z}\Big|_{z=0} = 1.$$

   (c) From the Taylor expansion of $\cos z$ we see that

$$\cos z - 1 = z^2(-\frac{1}{2} + \frac{z^2}{4!} + \cdots) =: z^2 h(z).$$

   Since $h(0) = -1/2 \neq 0$, $f(z) = \cos z/(\cos z - 1)$ has a pole of order 2 at 0. By rule 1a we get

$$\operatorname{Res}\left(\frac{\cos z}{\cos z - 1};0\right) = \left(\frac{\cos z}{h(z)}\right)'\Big|_{z=0} = \frac{-h(z)\sin z - h'(z)\cos z}{h(z)^2}\Big|_{z=0} = \frac{h'(0)}{h(0)^2} = 0,$$

   where the last step follows from the series defining $h$.

4. Let $f(z) = 3z^2$ and $g(z) = z^5 + 1$. For $|z| = 1$ we have

$$|f(z)| = 3|z|^2 = 3, \quad |g(z)| \leq |z|^5 + 1 = 2.$$

   Thus $|g(z)| < |f(z)|$ for $z$ on the boundary of $D(0,1)$. Rouche's theorem gives that $f$ and $f+g$ have tha same number of zeros in $D(0,1)$. $f$ has clearly 2 zeros in $D(0,1)$ so $z^5 + 3z^2 + 1$ also has 2 zeros in $D(0,1)$. As a degree 5 polynomial has 5 zeros in $\mathbb{C}$, $z^5 + 3z^2 + 1$ has 3 zeros outside of $D(0,1)$. (It is clear that there are no zeros on the boundary.)

5. Observe first that $f$ is a Möbius transformation so that it is a bijective map from the Riemann sphere to the Riemann sphere, takes circles/lines to circles/lines, and preserves angles. We have $f(0) = -1$, $f(i) = 0$, and $f(\infty) = 1$ and so the image of the imaginary axis must be the real axis.

   The image of the circle $|z| = 1$ must be a line since $f(-i) = \infty$, and since $f(i) = 0$ this line must go through the origin. Moreover, this line must intersect the real axis (=image of the imaginary axis) at right angle. Hence, $f(\{|z| = 1\}) =$ the imaginary axis.

To compute the image of the circle $C = \{|z+i| = 2\}$ we note that $f(-3i) = 2$ and $f(i) = 0$. Thus, $f(C)$ is a circle (since no point on $C$ is mapped to $\infty$) going through 2 and 0. This circle must intersect the real axis (= image of imaginary axis) at right angles, and so $f(C) = \{|z-1| = 1\}$.

Since $f$ preserves angles, in particular at the point $i$, it follows that

$$f(\Omega) = \{z;\, \operatorname{Re} z > 0,\, |z-1| > 1\}.$$

Draw the picture!

6. We want to compute

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x\, dx}{x^2+x+1} = \lim_{R\to\infty} \operatorname{Im} \int_{-R}^{R} \frac{e^{ix}\, dx}{x^2+x+1}.$$

Let $f(z) = e^{iz}/(z^2+z+1)$; the poles of $f$ are simple and located at $(-1 \pm i\sqrt{3})/2$. Let $\gamma_R$ be the line segment from $-R$ to $R$ and let $\Gamma_R$ be the upper half of the circle $|z| = R$ oriented counterclockwise.

On one hand we have

$$\begin{aligned}
\int_{\gamma_R+\Gamma_R} f(z)\, dz &= 2\pi i \operatorname{Res}(f; (-1+i\sqrt{3})/2) = 2\pi i \frac{e^{iz}}{2z+1}\Big|_{z=(-1+i\sqrt{3})/2} = \cdots \\
&= \frac{2\pi e^{-\sqrt{3}/2}}{\sqrt{3}}(\cos(1/2) - i\sin(1/2)),
\end{aligned}$$

by the Residue theorem and residue rule 2. On the other hand,

$$\int_{\gamma_R+\Gamma_R} f(z)\, dz = \int_{-R}^{R} \frac{e^{ix}\, dx}{x^2+x+1} + \int_{\Gamma_R} \frac{e^{iz}\, dz}{z^2+z+1}.$$

The second integral on the right-hand side goes to 0 as $R \to \infty$ since

$$\begin{aligned}
\left| \int_{\Gamma_R} \frac{e^{iz}\, dz}{z^2+z+1} \right| &\le \sup_{\Gamma_R} \frac{|e^{iz}|}{|z^2+z+1|} \pi R \\
&\le \sup_{\Gamma_R} \frac{e^{\operatorname{Re} iz}}{|z|^2-|z|-1} \pi R \\
&\le \sup_{\Gamma_R} \frac{e^{-\operatorname{Im} z}}{R^2-R-1} \pi R \\
&\le \frac{\pi R}{R^2-R-1} \to 0, \quad R \to \infty.
\end{aligned}$$

Hence,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}\, dx}{x^2+x+1} = \frac{2\pi e^{-\sqrt{3}/2}}{\sqrt{3}}(\cos(1/2) - i\sin(1/2)),$$

and so $I = -2\pi e^{-\sqrt{3}/2}\sin(1/2)/\sqrt{3}$.

7. The function $F(z) = z^3/3 - \cos z$ is a holomorphic primitive of $z^2 + \sin z$ in $\mathbb{C}$. By Prop. 3.4 in the lecture notes we thus get

$$\int_\gamma z^2 + \sin z \, dz = [F(z)]_{-i\pi/2}^{i\pi/2} = \cdots = -i\pi^3/12.$$

   Alternatively, one can use Cauchy's theorem to change the integration contour to the line segment from $-i\pi/2$ to $i\pi/2$. This is parametrized by $z = it$, $t \in [-\pi/2, \pi/2]$, and the resulting integral in $t$ is straightforward to compute.

8. We imitate the proof of Schwarz's lemma.

   Since $f$ has a zero of order 2 at 0, the singularity of $g(z) := f(z)/z^2$ is removable. Hence, $g$ is holomorphic in $D(0,1)$. Notice that since $f$ maps $D(0,1)$ to $D(0,1)$ we have $|f(z)| < 1$ for all $z \in D(0,1)$. Hence, for any $r < 1$ we have

$$\sup_{|z|=r} |g(z)| = \sup_{|z|=r} \frac{|f(z)|}{|z|^2} < \frac{1}{r^2}.$$

   By the maximum principle we then get

$$\sup_{|z|\le r} |g(z)| < \frac{1}{r^2}.$$

   Since this holds for any $r < 1$ we conclude that $|g(z)| \le 1$ for all $z \in D(0,1)$. Hence, $|f(z)| = |z|^2 |g(z)| \le |z|^2$ for all $z \in D(0,1)$.