Suggested solutions

Analytic function theory, MMG700, Exam Jan 3 2018

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1. See the literature.

2. See the literature.

3. (a) The derivative of φ is $1/(z+1)^2 \neq 0$. Hence φ is conformal.

   (b) The image of the real axis is either a line or a circle since φ is a Möbius transformation. Since $\varphi(x) \in \mathbb{R}$ if $x \in \mathbb{R}$ it follows that the image of the real axis is the real axis.

   The image of the unit circle is also either a line or a circle. Since -1 is on the unit circle and $\varphi(-1) = \infty$, the image of the unit circle must be a line. Moreover, $\varphi(i) = (1+i)/2$ and $\varphi(1) = 1/2$ and so it follows that the image of the unit circle is the line given by $\operatorname{Re} z = 1/2$.

   (c) See the literature.

4. For z with $|z| = 2$ we have

$$|4z^2 + z + 1| \leq 4 \cdot 2^2 + 2 + 1 = 19 < 32 = |z^5|.$$

   By Rouche's theorem z^5 and $z^5 + 4z^2 + z + 1$ have the same number of zeros in $D(0,2)$. Hence, $z^5 + 4z^2 + z + 1$ has 5 zeros in $D(0,2)$.

   For z with $|z| = 1$ we have

$$|z^5 + z + 1| \leq 3 < 4 = |4z^2|.$$

   Rouche's theorem therefore gives that $4z^2$ and $z^5 + 4z^2 + z + 1$ have the same number of zeros in $D(0,1)$. Thus, $z^5 + 4z^2 + z + 1$ has 2 zeros in $D(0,1)$.

   We conclude that z^5+4z^2+z+1 has $5-2 = 3$ zeros in the annulus $\{z;\ 1 < |z| < 2\}$.

5. We begin by doing a partial fraction decomposition:

$$\frac{1}{(z-1)(z-i)} = \frac{1+i}{2}\left(\frac{1}{z-1} - \frac{1}{z-i}\right).$$

   Next, since $|1/z| < 1$ if $|z| > 1$, we get

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-1/z} = \frac{1}{z}\sum_{k=0}^{\infty}(1/z)^k = \sum_{k=0}^{\infty} z^{-k-1}$$

   and

$$\frac{1}{z-i} = \frac{1}{z}\frac{1}{1-i/z} = \frac{1}{z}\sum_{k=0}^{\infty}(i/z)^k = \sum_{k=0}^{\infty} i^k z^{-k-1}$$

   with convergence in $\{z;\ |z| > 1\}$. Hence,

$$\frac{1}{(z-1)(z-i)} = \frac{1+i}{2}\sum_{k=0}^{\infty}(1-i^k)z^{-k-1}.$$

6. The function $z^2(1+z^2)$ has a double zero at $z=0$ and simple zeros at $z=\pm i$. $\tan(z/2)$ is holomorphic and $\neq 0$ in neighborhoods of these points, is $\neq 0$ at $z=\pm i$, and has a simple zero at $z=0$ since $\sin(z/2)$ has. It follows that f has poles of order 1 at $z=0$ and $z=\pm i$. Notice that these poles are in $D(0,2)$.

   Since $\tan(z/2)=\sin(z/2)/\cos(z/2)$ we see that f also has singularities where $\cos(z/2)$ has zeros, i.e., at $z=\pi+2k\pi$, $k\in\mathbb{Z}$. Since $(\cos(z/2))'=-\sin(z/2)/2$ and $\sin(z/2)\neq 0$ for $z=\pi+2k\pi$ it follows that f has poles of order 1 at $z=\pi+2k\pi$, $k\in\mathbb{Z}$. Notice that these points are not in $D(0,2)$.

   By Residue rule 1c we get

$$\operatorname{Res}(f;0)=\lim_{z\to 0} z\frac{\tan(z/2)}{z^2(1+z^2)}=\lim_{z\to 0}\frac{\sin(z/2)}{z}\frac{1}{(1+z^2)\cos(z/2)}=\frac{1}{2}$$

   and, by Residue rule 1a or 1c,

$$\operatorname{Res}(f;\pm i)=\frac{\tan(z/2)}{z^2(z\pm i)}\bigg|_{z=\pm i}=\cdots=\frac{1}{2}\frac{1-e}{1+e}.$$

   The Residue theorem therefore gives

$$\begin{aligned}\int_{|z|=2} f(z)\,dz &= 2\pi i\big(\operatorname{Res}(f;0)+\operatorname{Res}(f;i)+\operatorname{Res}(f;-i)\big)=2\pi i\left(\frac{1}{2}+\frac{1-e}{1+e}\right)\\ &= i\pi\frac{3-e}{1+e}.\end{aligned}$$

7. Set $\varphi(a)=\displaystyle\int_0^\infty \frac{\cos(ax)}{1+x^2}\,dx$. Since cos is an even function we have $\varphi(a)=\varphi(-a)$; it is therefore enough to compute $\varphi(a)$ for $a\geq 0$. We may notice that

$$\varphi(0)=\int_0^\infty \frac{1}{1+x^2}\,dx=[\arctan x]_0^\infty=\pi/2,$$

   but it will also follow below that $\varphi(0)=\pi/2$.

   Set $f(z)=\dfrac{e^{iaz}}{1+z^2}$, let $\Gamma_1(R)=[-R,R]$, and notice that

$$\int_0^\infty \frac{\cos(ax)}{1+x^2}\,dx=\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}\,dx=\operatorname{Re}\left(\lim_{R\to\infty}\frac{1}{2}\int_{\Gamma_1(R)} f(z)\,dz\right) \qquad (*)$$

   since $\cos(ax)/(1+x^2)$ is even and integrable on $\mathbb{R}$. Let $\Gamma_2(R)$ be the upper half of the circle given by $|z|=R$ oriented counterclockwise. For $R>1$ the cycle $\Gamma_1(R)+\Gamma_2(R)$ encloses the simple pole of f at $z=i$ but no other pole of f. We have

$$\operatorname{Res}(f;i)=\frac{e^{iaz}}{z+i}\bigg|_{z=i}=\frac{e^{-a}}{2i}$$

   and so the Residue theorem gives

$$\int_{\Gamma_1(R)} f(z)\,dz+\int_{\Gamma_2(R)} f(z)\,dz=2\pi i\frac{e^{-a}}{2i}=\pi e^{-a}.$$

For z on $\Gamma_2(R)$ we have $\operatorname{Im} z \geq 0$ and so, since $a \geq 0$, $|e^{iaz}| = e^{\operatorname{Re}(iaz)} = e^{-a\operatorname{Im} z} \leq 1$. Therefore,

$$\left| \int_{\Gamma_2(R)} f(z)\, dz \right| \leq \pi R \sup_{z \in \Gamma_2(R)} \left| \frac{e^{iaz}}{1+z^2} \right| \leq \frac{\pi R}{1+R^2} \to 0$$

as $R \to \infty$. Hence,

$$\lim_{R\to\infty} \int_{\Gamma_1(R)} f(z)\, dz = \pi e^{-a}$$

and so, by $(*)$,

$$\int_0^\infty \frac{\cos(ax)}{1+x^2}\, dx = \operatorname{Re} \frac{\pi e^{-a}}{2} = \frac{\pi e^{-a}}{2}$$

for $a \geq 0$. We conclude that $\varphi(a) = \pi e^{-|a|}/2$ for any $a \in \mathbb{R}$.

8. Set $f(z) := \dfrac{\partial u}{\partial z} = (u'_x - iu'_y)/2$. Since u is harmonic,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial^2 u}{\partial z \partial \bar{z}} = \Delta u/4 = 0,$$

and so f is holomorphic. Thus, f^2 is holomorphic. Since

$$f^2 = (u'_x - iu'_y)^2/4 = \big((u'_x)^2 - (u'_y)^2 - 2iu'_x u'_y\big)/4$$

and since the real and imaginary part of a holomorphic function is harmonic it follows that $(u'_x)^2 - (u'_y)^2$ and $u'_x u'_y$ are harmonic.