Suggested solutions

Analytic function theory, MMG700, Exam 27 October 2017

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1. See the literature.

2. See the literature.

3. See the literature for 3a. For 3b, notice that $\left(\frac{1}{z-1}\right)' = -\frac{1}{(z-1)^2}$. Thus, by part (a), we get

$$\int_{\mathcal{C}} \frac{dz}{(z-1)^2} = -\int_{\mathcal{C}} \left(\frac{1}{z-1}\right)' dz = -\Big(\frac{1}{z-1}\Big|_{z=1+i} - \frac{1}{z-1}\Big|_{z=0}\Big) = -1+i.$$

4. Set $f(z) = z^4$ and $g(z) = z^3 + 5$. If $|z| = 2$ we have $|f(z)| = 2^4 = 16$ and $|g(z)| \leq |z^3| + 5 = 2^3 + 5 = 13$. Hence, $|f(z)| > |g(z)|$ if $|z| = 2$ and so it follows by Rouche's theorem that z^4 and $z^4 + z^3 + 5$ have the same number of zeros in $D(0,2)$. Thus, $z^4 + z^3 + 5$ has 4 zeros in $D(0,2)$.

   On the other hand, if $|z| = 1$, then $|f(z)| = 1$ and $|g(z)| \geq 5 - |z|^3 = 5 - 1 = 4$ so $|g(z)| > |f(z)|$ if $|z| = 1$. By Rouche's theorem z^3+5 and z^4+z^3+5 have the same number of zeros in $D(0,1)$. Since the zeros of z^3+5 have absolute value $5^{1/3} > 1$ it follows that $z^4 + z^3 + 5$ has no zeros in $D(0,1)$. Hence, $z^4 + z^3 + 5$ has 4 zeros in the annulus $\{1 < |z| < 2\}$.

5. We notice first that
$$\int_0^\infty \frac{\cos 2x}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos 2x}{(x^2+1)^2}$$
   since the integrand is an even function on $\mathbb{R}$. To compute the integral on the right-hand side set $f(z) = e^{2iz}/(z^2+1)^2$, let γ_R be the interval $[-R, R]$, and let Γ_R be the top half of the circle $\partial D(0,R)$ oriented counterclockwise. f is holomorphic in $\mathbb{C} \setminus \{\pm i\}$ and has poles of order 2 at $\pm i$ since $(z^2+1)^2 = (z-i)^2(z+i)^2$; the residue at i is, by Residue computation 1a,
$$\left(\frac{e^{2iz}}{(z+i)^2}\right)'\Big|_{z=i} = \frac{2ie^{-2}(2i)^2 - e^{-2}2(2i)}{(2i)^4} = \ldots = -3ie^{-2}/4.$$
   Only i is in the set bounded by $\Gamma_R + \gamma_R$ so Cauchy's residue theorem gives
$$\int_{\gamma_R} f(z)dz + \int_{\Gamma_R} f(z)dz = 2\pi i \mathrm{Res}(f; i) = 3\pi e^{-2}/2.$$
   The second integral on the left-hand side goes to 0 as $R \to \infty$ since
$$\left|\int_{\Gamma_R} f(z)dz\right| \leq \sup_{z\in\Gamma_R}\left|\frac{e^{2iz}}{(z^2+1)^2}\right| \cdot \ell(\Gamma_R) \leq \sup_{z\in\Gamma_R} \frac{e^{-2\mathrm{Im}\ z}}{(|z|^2+1)^2}\cdot \pi R \leq \frac{\pi R}{(R^2+1)^2}.$$

Hence,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos 2x + i\sin 2x}{(x^2+1)^2}dx = \lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 3\pi e^{-2}/2,$$

and so

$$\int_0^\infty \frac{\cos 2x}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos 2x}{(x^2+1)^2} = \frac{1}{2}\mathrm{Re}\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos 2x + i\sin 2x}{(x^2+1)^2}dx = 3\pi e^{-2}/4.$$

6. To find the fixed points we solve the equation $z = \varphi(z)$, i.e., $z(z - i\sqrt{2}) = 1$. By completing the square this equation may be written as $(z - i/\sqrt{2})^2 = 1/2$, which has the solutions $(i \pm 1)/\sqrt{2}$.

   For part (b) notice first that the circle $C := \partial D(i/\sqrt{2}, 1/\sqrt{2})$ goes through the two fixed points, the origin, and the point $i\sqrt{2}$. Since φ is a Möbius transformation mapping $i\sqrt{2}$ to ∞ it follows that φ maps C to the straight line trough the two fixed points. Moreover, since $\varphi(0) = i/\sqrt{2}$ it follows that the lower half of C is mapped to the line segment starting at $(-1+i)/\sqrt{2}$ and ending at $(1+i)/\sqrt{2}$.

   The line through the two fixed points is mapped to a circle going through the two fixed points and since $\varphi(i/\sqrt{2}) = i\sqrt{2} \in C$ the line segment between the two fixed points is mapped to the upper half of C. Since φ is conformal and bijective on the Riemann sphere it follows that the lower part of the disc bounded by C is mapped to the upper part of the same disc.

7. Let Ω_1 be the given set and let $f_1(z) = z^3$. Then f_1 is conformal in Ω_1 and maps it to the upper half of the unit disc, that we denote by Ω_2.

   Let $f_2(z) = \dfrac{1+z}{1-z}$. Since f_2 is a Möbius transformation such that $f_2(-1) = 0$, $f_2(0) = 1$, $f_2(1) = \infty$, and $f_2(i) = i$ it follows that $f_2(\Omega_2)$ is the first quarter, which we denote by Ω_3.

   Setting $f_3(z) = z^2$ we get that $f_3(\Omega_3)$ is the upper half-plane. The upper half-plane is mapped to the unit disc by $f_4(z) = (z-i)/(z+i)$. The composition $f_4 \circ f_3 \circ f_2 \circ f_1(z)$ has the desired properties.

8. We have, e.g., by the Residue theorem, that

$$f'(0) = \frac{1}{2\pi i}\int_{\partial D(0,1)}\frac{f(z)}{z^2}dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{it})}{e^{2it}}ie^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-it}dt.$$

   Hence,

$$\overline{f'(0)} = \frac{1}{2\pi}\int_0^{2\pi}\overline{f(e^{it})}e^{it}dt = \frac{1}{2\pi i}\int_0^{2\pi}\overline{f(e^{it})}ie^{it}dt = \frac{1}{2\pi i}\int_{\partial D(0,1)}\overline{f(z)}\,dz.$$