Solution Exam Fourier Analysis

2018-08-23

1. (a) Integrate (1), $\int_0^x f(t)\,dt$:

$$\int_0^x (\pi - t)\,dt = 2\int_0^x \sum_1^\infty \frac{\sin nt}{n}\,dt = 2\sum_1^\infty \frac{-\cos nx}{n^2} + C$$

$$\pi x - \frac{1}{2}x^2 = -2\sum_1^\infty \frac{\cos nx}{n^2} + C.$$

Integrate again, $\int_0^{2\pi} dx$:

$$\pi 4 \cdot \frac{1}{2}\pi^2 - \frac{8}{6}\pi^3 = 2\pi C, \qquad \frac{1}{3}\pi^3 = \pi C, \quad C = \frac{1}{3}\pi^2.$$

$$-2\sum_1^\infty \frac{\cos nx}{n^2} = \pi x - \frac{1}{2}x^2 - \frac{1}{3}\pi^2,$$

$$\sum_1^\infty \frac{\cos nx}{n^2} = -\frac{1}{2}\pi x + \frac{1}{4}x^2 + \frac{1}{6}\pi^2, \quad 0 < x < 2\pi$$

Let $x \to 2x$, $0 < x < \pi \to 0 < x < \frac{\pi}{2}$,

$$\sum_1^\infty \frac{\cos 2nx}{n^2} = -\pi x + x^2 + \frac{1}{6}\pi^2.$$

(b) Take $x = 0$, in the above series which is convergent with the sum being continuous and in PC',

$$\sum_1^\infty \frac{1}{n^2} = \frac{1}{6}\pi^2$$

2. We use Fourier series as orthogonal expansion

(a) ~~$2\sin^2 x = 1 - \cos 2x$, $2\sin^2 x \cos x = \cos x$~~

$$2\sin^2 x \cos x = \sin 2x \sin x = \frac{1}{2}(\cos x - \cos 3x)$$

$\cos 3x \perp \cos x$ in $L^2(-\pi, \pi)$

$\Rightarrow \{\cos x, \cos 3x\}$ is an orthogonal basis of V

(b) $f(x) = \cos^2 2x = \frac{1}{2}(1 + \cos 4x)$

f is orthogonal to V, $\text{dist}(f, V) = \|f\|$

$\|f\|^2 = \frac{1}{4}(2\pi + \pi) = \frac{3}{4}\pi$, $\text{dist}(f\ V) = \frac{\sqrt{3}}{2}\pi$.

3. $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, $\frac{1}{x^2+1}$ is even

$\cos 2x \sin x = \frac{1}{2}(\sin 3x - \sin x)$, $\frac{1}{x}$ is odd

(a) $\mathcal{F}: \frac{1}{1+x^2} \to \pi e^{-|\xi|}$,

$$\frac{\cos x}{1+x^2} \to \frac{1}{2}\pi\left(e^{-|\xi+1|} + e^{-|\xi-1|}\right).$$

$\mathcal{F}: \frac{\sin x}{x} \to \pi \chi_1(\xi)$

$$\frac{\sin 3x}{x} \to \pi \chi_3(\xi)$$

$$\frac{\cos 2x \sin x}{x} \to \frac{1}{2}\pi(\chi_3(\xi) - \chi_1(\xi))$$

(b) $(f, g) = \frac{1}{2\pi}(\hat{f}, \hat{g}) = \left(\frac{1}{2}\pi\right)^2 \frac{1}{2\pi}\int_{\mathbb{R}} (\chi_3(\xi) - \chi_1(\xi))\left(e^{-|\xi+1|} + e^{-|\xi-1|}\right).$

$\chi_3(\xi) - \chi_1(\xi) = \chi_{[-3,-1]}(\xi) + \chi_{[1,3]}(\xi)$

$\int \chi_{[-3,-1]}(\xi) e^{-|\xi+1|} d\xi = \int_{-3}^{-1} e^{\xi+1} d\xi = e(e^{-1} - e^{-3}).$

$$\int \chi_{[1,3]}(\xi)\, e^{-|\xi+1|}\, d\xi = -\left(e^{-4} - e^{-2}\right) = e^{-2} - e^{-4}$$

$$\Rightarrow \int \left(\chi_3(\xi) - \chi_1(\xi)\right) e^{-|\xi+1|}\, d\xi = 1 - e^{-2} + e^{-2} - e^{-4} = 1 - e^{-4}$$

Change variables, $\xi \to -\xi$:

$$\int \left(\chi_3(\xi) - \chi_1(\xi)\right) e^{-|\xi-1|}\, d\xi = 1 - e^{-4}.$$

Answer: $(f, g) = \frac{\pi}{2^3} \cdot 2\left(1 - e^{-4}\right) = \frac{\pi}{2^2}\left(1 - e^{-4}\right)$,

$$\|f\|^2 = \frac{1}{2\pi}\|\hat{f}\|^2 = \frac{1}{2\pi}\left(\frac{\pi}{2}\right)^2 \int_{\mathbb{R}} \left(e^{-|\xi+1|} + e^{-|\xi-1|}\right)^2 d\xi$$

$$= \frac{\pi}{2^3} \int_{\mathbb{R}} \left(e^{-2|\xi+1|} + 2e^{-|\xi+1|-|\xi-1|} + e^{-2|\xi-1|}\right) d\xi.$$

$$= \frac{\pi}{2^3}\left(2\int_{\mathbb{R}} e^{-2|\eta|}\, d\eta + 2\int_{\mathbb{R}} e^{-|\xi+1|-|\xi-1|}\, d\xi\right), \quad \left(\text{since } \int_{\mathbb{R}} e^{-2|\xi+a|}\, d\xi = \int_{\mathbb{R}} e^{-2|\xi|}\, d\xi.\right)$$

$$\int_{\mathbb{R}} e^{-|\xi+1|-|\xi-1|}\, d\xi = \int_{-\infty}^{-1} e^{\xi+1+(\xi-1)}\, d\xi + \int_{-1}^{1} e^{-(\xi+1)+(\xi-1)}\, d\xi + \int_{1}^{\infty} e^{-(\xi+1)-(\xi-1)}\, d\xi$$

$$= \frac{1}{2}e^{-2} + 2e^{-2} + \frac{1}{2}e^{-2} = 3e^{-2}.$$

$$\cancel{\frac{(f, g)}{\|f\|\|g\|} = \frac{\pi}{2^2}\left(1 - e^{-4}\right).}$$

$$\|f\|^2 = \frac{\pi}{2^3}\left(2+6e^{-2}\right) = \frac{\pi}{2^2}(1+3e^{-2}).$$

$$\|g\|^2 = \frac{1}{2\pi}\left(\frac{1}{2}\pi\right)^2 \int_{\mathbb{R}} (\chi_3 - \chi_1)^2(\xi)\,d\xi = \frac{\pi}{2^3}\cdot 4 = \frac{\pi}{2}$$

$$\frac{(f,g)}{\|f\|\|g\|} = \frac{\cancel{\pi}}{2^{\cancel{3}}}(1-e^{-4})\frac{\cancel{2}}{(1+3e^{-2})^{1/2}\sqrt{\cancel{\pi}}}\cdot\frac{\sqrt{2}}{\sqrt{\cancel{\pi}}}$$

$$= \frac{1-e^{-4}}{\sqrt{2}\sqrt{1+3e^{-2}}}.$$

Answer: The angel $\angle f\&g = \arccos\dfrac{1-e^{-4}}{\sqrt{2}(1+3e^{-2})^{1/2}}$.

4. Homogeneous eq. with homog. boundary value

$$u(x,t) = \sum_1^\infty \sin nx\,\left(a_n\cos(nct) + b_n\sin(nct)\right)$$

$$0 = u_t(x,0) = \sum_1^\infty \sin nx\; b_n(nc) \Rightarrow b_n = 0.$$

$$f(x) = u(x,0) = \sum_1^\infty \sin nx\; a_n, \Rightarrow a_n = \frac{2}{n}.$$

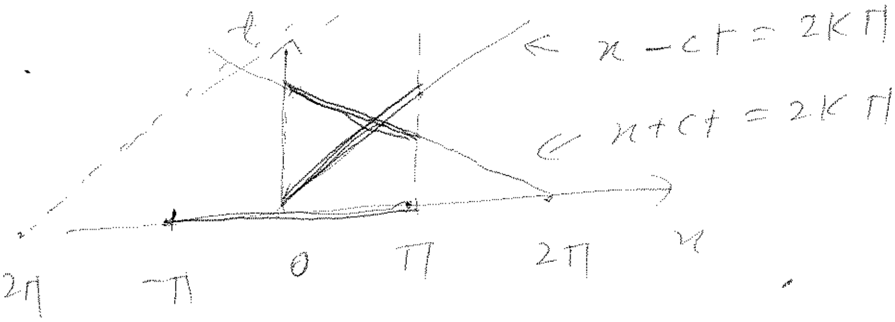
Answer: $u(x,t) = \displaystyle\sum_1^\infty \frac{2}{n}\sin nx\cos nct$

$$u(x,t) = \sum_1^\infty \frac{2}{n}\left(\sin n(x+ct) + \sin n(x-ct)\right)$$

$$= \sum_1^\infty \frac{2}{n}\sin n(x+ct) + \sum \frac{2}{n}\sin n(x-ct),$$

each of the series is the periodic extension of $\pi - x$, $0<x<2\pi$ with discontinuity at 0 & 2π, ie. discontinuous at all $2k\pi$.

Answer: Discontinuous at $\boxed{x \pm ct = 2k\pi}$



5. $u'' - 2u' - 3u = H(t-1)$

$\mathcal{L}$: $U(z) = \mathcal{L}(u)(z)$

$$z^2 U(z) - 2z U(z) - 3U(z) = e^{-z}\frac{1}{z}$$

$$U(z)\left(z^2 - 2z - 3\right) = e^{-z}\frac{1}{z} \qquad U(z) = e^{-z}\frac{1}{z}\,\frac{1}{z^2 - 2z - 3}$$

$$U(z) = e^{-z}\frac{1}{z(z-3)(z+1)} = e^{-z}\left(-\frac{1/3}{z} + \frac{1/4}{z+1} + \frac{1/12}{z-3}\right)$$

$$u(t) = H(t-1)\left(-\frac{1}{3} + \frac{1}{4}e^{-(t-1)} + \frac{1}{12}e^{3(t-1)}\right).$$

The discontinuity appears at $t = 1$.