Fourier analysis. Additional exercises.

1. Consider the wave equation $u''_{tt} = c^2 u''_{xx}$. By d'Alemberts solution (Folland Ex. 1.1.6), any solution is of the form $u(x,t) = f(x+ct) + g(x-ct)$.

   (a) Show that $u(x,t) = \sin(x)\sin(ct)$ is a solution of the wave equation, and write it in the form of d'Alemberts solution.

   (b) Using d'Alemberts method, find a solution to the wave equation $u''_{tt} = 4u''_{xx}$, valid for all x and t, such that $u(x,0) = x^2$, $u'_t(x,0) = x$.

2. Consider the vibrating string with fixed endpoints, $u(0,t) = u(\pi,t) = 0$. Suppose that the initial conditions are $u(x,0) = \sin(2x)$, $u'_t(x,0) = 3\sin(5x)$. What is the solution $u(x,t)$?

3. If you know the complex Fourier coefficients of f, what can you say about the Fourier coefficients of $f(x-a)$ and of $e^{ikx}f(x)$ (where a is real and k is integer)?

4. When f and g are 2π-periodic Riemann integrable functions, define their convolution by

$$(f * g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)g(x-y)\,dy.$$

   Denoting Fourier coefficients by $c_n(f)$, show that $c_n(f*g) = c_n(f)c_n(g)$.

5. Let f be the 2π-periodic function defined by $f(x) = e^{\cos(x^2)}$ for $0 \le x < 2\pi$. What is the value of its Fourier series at $x = 4\pi$?

6. Find numbers c_n such that

$$\sum_{n=1}^{\infty} c_n \sin(nx) = \begin{cases} x, & 0 < x < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$

   What is the sum of the series for $x = \pi/2$?

7. Determine the Fourier series in real form of the 2π-periodic function that equals $x(x^2-\pi^2)$ in $[-\pi,\pi]$. What is the sum of the series at the points 2π and $3\pi/2$?

8. Let f be the 2π-periodic function defined by $f(x) = \cosh(x) = (e^x + e^{-x})/2$ for $|x| \le \pi$. Express it as a complex Fourier series. Compute

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1}.$$

9. Let f be the 2π-periodic function defined by $f(x) = \cos(ax)$ for $|x| \le \pi$, where a is not an integer. Express it as a complex Fourier series. Deduce the identity

$$\pi\cot(\pi a) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{a+n}, \qquad a \notin \mathbb{Z}.$$

10. Show that

$$\frac{\sin x}{x} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(nx), \qquad 0 < x < \pi,$$

where

$$b_n = \frac{1}{\pi} \int_{(n-1)\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx.$$

Use this result to compute

$$\int_0^{\infty} \frac{\sin x}{x}\, dx.$$

11. Expand the function $\cos x$ as a sine series on the interval $(0, \pi)$. Use the result to compute

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}.$$

12. Solve the heat conduction problem

$$\begin{cases} u'_t = 3u''_{xx}, & 0 \le x \le \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \ge 0, \\ u(x,0) = \sin x \cos(4x), & 0 \le x \le \pi. \end{cases}$$

13. Solve the problem

$$\begin{cases} u'_t = 2u''_{xx}, & t > 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = 1, & 0 < x < \pi. \end{cases}$$

14. Solve the problem

$$\begin{cases} u'_t = 2u''_{xx}, & t > 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = \cos(3x), & 0 < x < \pi. \end{cases}$$

15. Solve the problem

$$\begin{cases} u'_t = u''_{xx}, & t > 0, \quad 0 < x < \pi, \\ u'_x(0,t) = u'_x(\pi,t) = 0, & t > 0, \\ u(x,0) = 1, & 0 < x < \pi/2, \\ u(x,0) = 0, & \pi/2 < x < \pi. \end{cases}$$

16. Solve the problem

$$u'_t = u''_{xx}, \qquad u'_x(0,t) = u'_x(\pi,t) = 0, \quad u(x,0) = \cos\left(\frac{x}{2}\right),$$

in the region $t > 0$, $0 < x < \pi$.

17. A large steak with thickness l and heat diffusivity a is to be cooked in an oven with temperature T. It is taken directly from the fridge, where the temperature is 0. The temperature inside the steak can then be described by

$$\begin{cases} u'_t = au''_{xx}, & 0 \le x \le l, \quad t > 0, \\ u(0,t) = u(l,t) = T, & t > 0, \\ u(x,0) = 0, & 0 \le x \le l. \end{cases}$$

(a) Determine, as a series, $u(x,t)$. Since the boundary conditions are inhomogeneous, one should shift the temperature scale so that T corresponds to 0 (equivalently, introduce the function $v(x,t) = u(x,t) - T$).

(b) Suppose all terms in the series except the first can be ignored. When will the steak be cooked if that happens when the temperature is everywhere at least $T/4$? How much longer does it take if the thickness of the steak is doubled?

18. Addendum to Folland, Exercise 2.5.5: Where should the string be plucked if we do not wish to hear the sixth overtone (corresponding to the seventh term in the Fourier series)? Motivation: The first five overtones are close to tones in the usual scale. The sixth overtone is somewhere between two half-tones, and thus sounds false (at least to people accustomed to European music).

19. Addendum to Folland, Exercise 2.5.6: Can we choose δ so the sixth overtone is avoided?

20. Let $f(t) = 1 - t^2$ for $|t| \le 1$ and let f be 2-periodic. Determine a bounded solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x > 0, \quad -\infty < t < \infty, \\ u(0,t) = f(t), & -\infty < t < \infty. \end{cases}$$

21. Solve the Laplace equation $\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$ in the annulus $1 < r < 2$ (polar coordinates), with boundary values $u(1,\theta) = 0$,

$$u(2,\theta) = 1 - \frac{\theta^2}{\pi^2} \quad \text{for} \quad |\theta| \le \pi.$$

22. Solve the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = y, & 0 < x < 2,\ 0 < y < 1, \\ u(x,0) = 0, & u(x,1) = 0, \\ u(0,y) = y - y^3, & u(2,y) = 0. \end{cases}$$

23. Solve the problem

$$\begin{cases} \sqrt{1+t}\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 1, & u(1,t) = 0, \\ u(x,0) = 1 - x^2. \end{cases}$$

24. Solve the problem

$$\begin{cases} u''_{xx} + u''_{yy} + 20u = 0, & 0 < x < 1, \quad 0 < y < 1, \\ u(0,y) = u(1,y) = 0, \\ u(x,0) = 0, & u(x,1) = x^2 - x. \end{cases}$$

25. Solve the inhomogeneous problem

$$\begin{cases} u'_t = 2u''_{xx} + \cos x, & 0 < x < \pi, \quad t > 0 \\ u'_x(0,t) = u'_x(\pi,t) = 0, & t > 0, \\ u(x,0) = 1, & 0 < x < \pi. \end{cases}$$

26. Solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = t \sin \pi x, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = \sin x. \end{cases}$$

27. Solve the problem (for $0 < x < \pi$ and $t > 0$)

$$\begin{cases} u'_t = (t+1)u''_{xx}, \\ u(0,t) = 0, \qquad u(\pi,t) = 1, \qquad u(x,0) = 0. \end{cases}$$

28. Solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = t + 1, \qquad u(1,t) = 0, \\ u(x,0) = 1 - x. \end{cases}$$

29. Solve the problem

$$\begin{cases} u_{xx} + 1 = \frac{1}{4} u_{tt}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = 0, \quad u(x,0) = x - x^2, \\ u(2,t) = -2, \quad u_t(x,0) = 0. \end{cases}$$

30. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad \sqrt{x^2 + y^2} < 1,$$

with boundary values $f(\theta) = \sin^2\theta + \cos\theta$ (in polar coordinates).

31. Let c_n be the coefficients in the Fourier series

$$e^{x^2} = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad 0 < x < 2\pi.$$

Is it true or false that

$$2xe^{x^2} = \sum_{n=-\infty}^{\infty} inc_n e^{inx}, \quad 0 < x < 2\pi?$$

32. Using known facts on Fourier series, find all 2π-periodic and twice continuously differentiable functions u such that $u''(x) = u(x + \pi)$ for all x.

33. Find a 1-periodic solution to the equation

$$x'' + 2x' + x = \{t\},$$

where $\{t\}$ is the fractional part of t (that is, the 1-periodic function defined by t for $0 \leq t < 1$). Give the answer in real form.

34. The function $f(x)$ is 2-periodic, and $f(x) = (x + 1)^2$ for $-1 < x < 1$. Determine a 2π-periodic solution to the equation

$$2y'' - y' - y = f(x).$$

35. The function $f(t)$ is 3-periodic, and

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 1, & 1 < t < 2, \\ 3 - t, & 2 \leq t \leq 3. \end{cases}$$

Determine, as a Fourier series, a periodic solution to

$$y'' + 3y = f(t).$$

36. Show that, in an inner product space, if $u_n \to u$ and $v_n \to v$ (in norm), then $\langle u_n, v_n \rangle \to \langle u, v \rangle$.

37. Show that, if $(e_k)_{k=1}^{\infty}$ is a complete orthonormal system in an inner product space, then

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle.$$

38. Find an orthonormal basis for the space of first degree polynomials, with inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

39. Use the previous exercise to determine the constants a, b that minimize the integral

$$\int_0^1 |e^x - ax - b|^2\, dx.$$

40. Determine the solution $y(x)$ to $y'' - y = 0$ which minimizes $\displaystyle\int_{-1}^{1} [1 + x - y(x)]^2 dx$.

41. Determine the polynomial $P(x)$ of degree at most 2 that minimizes
(a) $\displaystyle\int_0^{\infty} [\sqrt{x} - P(x)]^2 e^{-x} dx$, (b) $\displaystyle\int_0^{\infty} [e^{x/4} - P(x)]^2 x e^{-x} dx$.

42. Determine the polynomial of the form $P(x) = x^2 + ax + b$ that minimizes $\displaystyle\int_0^1 [P(x)]^2 dx$.

43. Let

$$Q_n(x) = \frac{d^n}{dx^n} \left(x^n (1 - x)^n \right), \quad n = 0, 1, 2, \ldots.$$

(Up to a change of variables, these are called *Legendre polynomials*.)

(a) Prove that

$$\int_0^1 f(x) Q_n(x)\, dx = (-1)^n \int_0^1 f^{(n)}(x)\, x^n (1 - x)^n\, dx$$

for sufficiently differentiable functions f.

(b) Show that $Q_n(x)$ and $Q_m(x)$ are orthogonal in $L^2(0,1)$ if $n \neq m$.

(c) Determine the norm $\|Q_n\|$ of Q_n in $L^2(0,1)$.

44. Find numbers a and b such that the integral

$$\int_0^{2\pi} \left| e^x - a e^{ix} - b e^{-ix} \right|^2\, dx$$

is minimized. Also compute the minimal value of the integral.

45. Apply Parseval's formula to the 2π-periodic function $f(x) = x$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^2}$.

46. Apply Parseval's formula to the 2π-periodic function $f(x) = x^2$, $|x| < \pi$. Use the result to compute $\sum_1^\infty \frac{1}{n^4}$.

47. Prove that

$$x(\pi - x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)^3}, \qquad 0 < x < \pi,$$

and use the result to compute $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$.

48. Find the complex Fourier coefficients of the function $(\cos x)^n$, with n a non-negative integer. Use the result to compute $\sum_{k=0}^{n} \binom{n}{k}^2$.

49. Define $J_n(x)$ through the Fourier series

$$e^{ix\sin(t)} = \sum_{n=-\infty}^{\infty} J_n(x) e^{int}$$

(they are called Bessel functions). Compute, for $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} |J_n(x)|^2.$$

50. Solve the heat conduction problem

$$\begin{cases} u'_t = u''_{xx}, & 0 \le x \le \frac{\pi}{2}, \quad t > 0, \\ u(0,t) = u'_x(\pi/2, t) = 0, & t \ge 0, \\ u(x,0) = x, & 0 \le x \le \frac{\pi}{2} \end{cases}$$

51. Determine all eigenvalues and eigenfunctions for the Sturm–Liouville problem

$$\begin{cases} f'' + \lambda f = 0, & 0 < x < a, \\ f(0) - f'(0) = 0, & f(a) + 2f'(a) = 0. \end{cases}$$

52. Determine all eigenvalues and eigenfunctions for the Sturm–Liouville problem

$$\begin{cases} -e^{-4x} \dfrac{d}{dx}\left(e^{4x} \dfrac{du}{dx}\right) = \lambda u, \quad 0 < x < 1, \\ u(0) = 0, \qquad u'(1) = 0. \end{cases}$$

Expand the function e^{-2x} as a series in the eigenfunctions.

53. A metal thread is bent into a circle. The ends are attached so that they are partially, but not completely, insulated from each other. The corresponding heat transfer problem can be modeled by

$$u_t' = k u_{xx}'', \qquad 0 < x < 1, \quad t > 0, \qquad u_x(0) = u_x(1) = \alpha\big(u(0) - u(1)\big),$$

where α and k are positive constants. Looking for separated solutions, $u(x,t) = X(x)T(t)$, one finds that X satisfies the Sturm–Liouville problem

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = X'(1) = \alpha\big(X(0) - X(1)\big).$$

   (a) Prove that the problem is symmetric in the sense that $\langle f'', g\rangle_{L^2([0,1])} = \langle f, g''\rangle_{L^2([0,1])}$, when f and g are sufficiently differentiable and satisfy the boundary conditions.
   (b) Prove that the problem has a non-trivial solution when $\lambda = 4n^2\pi^2$, $n \in \mathbb{Z}$. Prove that there are infinitely many other values of λ for which the problem also has a non-trivial solution.

54. Using the table in Folland's book, compute the Fourier transform of

   (a) $x\,\chi_{[-1,1]}(x)$,
   (b) $\sin x\,\chi_{[-\pi,\pi]}(x)$,
   (c) $e^{-x}H(x)$,
   (d) $e^{-|x|}\cos(x)$,
   (e) $\dfrac{1}{x^2+6x+13}$,
   (f) $\dfrac{x}{(x^2+1)^2}$,
   (g) $\dfrac{1}{(t^2+1)^2}$.

Here, H is Heaviside's function and $\chi_{[a,b]}(x)$ is the characteristic function of $[a,b]$, that is, $\chi_{[a,b]}(x) = 1$ for $x \in [a,b]$ and 0 else.

55. Use Fourier transform to compute, for $a \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\,dx.$$

56. If $f(x)$ has Fourier transform $\hat{f}(\xi)$, what is then the Fourier transform of $\cos(x)f(2x+1)$?

57. Complete the proof of (a) and (b) in Folland, Theorem 7.5.

58. Find a function u such that

$$\int_{-\infty}^{\infty} u(x-y)e^{-|y|}\,dy = e^{-x^4}.$$

59. For a and b positive and with $g_a(x) = 1/(x^2 + a^2)$, compute

$$\int_{-\infty}^{\infty} |(g_a * g_b)(x)|^2 \, dx.$$

60. Use Fourier transform to compute

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2(x^2+4)^2}\, dx,$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{(x^2+2)^2}{(x^4+4)^2}\, dx,$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2}\, dx,$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x-1)\sin(2x)}{(x-1)x}\, dx,$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx.$

61. With $f(x) = \sin(\pi x)/(x^2 - 1)$, show that

$$\hat{f}(\xi) = \begin{cases} \pi i \sin(\xi), & |\xi| \le \pi, \\ 0, & |\xi| > \pi. \end{cases}$$

Use this to compute

$$\int_{-\infty}^{\infty} \frac{\sin^2(\pi x)}{(x^2-1)^2}\, dx.$$

62. The function $f(t)$ has Fourier transform $\hat{f}(\omega) = \frac{\omega}{1+\omega^4}$. Compute

a) $\displaystyle\int_{-\infty}^{\infty} t f(t) dt,$ b) $f'(0)$.

63. The function $f(t)$ has Fourier transform $\dfrac{1}{|\omega|^3 + 1}$. Compute $\displaystyle\int_{-\infty}^{\infty} |f * f'|^2 dt.$

64. Determine the Fourier transform of the function

$$f(t) = \int_0^2 \frac{\sqrt{\omega}}{1+\omega} e^{i\omega t} d\omega.$$

Compute a) $\displaystyle\int_{-\infty}^{\infty} f(t)\cos t dt,$ b) $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt.$

65. Let $f(t) = \displaystyle\int_0^1 \sqrt{\omega}\, e^{\omega^2} \cos \omega t \, d\omega$. Compute $\displaystyle\int_{-\infty}^{\infty} |f'(t)|^2 dt.$

66. The continuous function $f(x)$ has Fourier transform $\widehat{f}(\xi) = \frac{\ln(1+\xi^2)}{\xi^2}$. Determine $f(0)$ and $\int_{-\infty}^{\infty} f(x)\,dx$.

67. Let $\phi_n(x)$ denote the function that equals 1 for $x \in [n - \frac{1}{2}, n + \frac{1}{2}]$ and 0 otherwise. Prove that $(\hat{\phi}_n)_{n=-\infty}^{\infty}$ is an orthogonal system in $L^2(\mathbb{R})$. Prove that it is not complete.

68. Show that the functions $\varphi_n(x) = \dfrac{\sin \frac{x}{2}}{\pi x} e^{inx}$, $n \in \mathbb{Z}$, are pairwise orthogonal in $L^2(\mathbf{R})$. Determine numbers c_n such that

$$\int_{-\infty}^{\infty} \left| \frac{1}{1+x^2} - \sum_{n=-N}^{N} c_n \varphi_n(x) \right|^2 dx$$

is minimal. Is the orthogonal system $(\varphi_n)_{n\in\mathbb{Z}}$ complete?

69. Let $\phi_n(x) = \sin(nx)$ for $0 < x < \pi$ and $\phi_n(x) = 0$ else. Compute values of c_n which minimize the integral

$$\int_{-\infty}^{\infty} \left| \frac{\sin \xi}{\xi} - \sum_{n=1}^{\infty} c_n \hat{\phi}_n(\xi) \right|^2 d\xi.$$

Also compute the minimum value.

70. Find (as an integral) a solution to the heat equation $u_{xx} = u_t$ for $t > 0$, $x \in \mathbb{R}$, where $u(x,0) = 1$ for $|x| < 1$ and 0 else.

71. Find a bounded solution to

$$\begin{cases} u_t = k u_{xx}, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = (1-2x^2)e^{-x^2}, & -\infty < x < \infty. \end{cases}$$

72. Find a bounded harmonic function $u(x,y)$ in the upper half-plane with boundary values

$$u(x,0) = \begin{cases} 1, & |x| < 1, \\ 0, & \text{else.} \end{cases}$$

73. Show that, if f is even, then under suitable conditions

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(\xi) \cos(\xi x)\, d\xi,$$

where

$$F(\xi) = \int_0^{\infty} f(x) \cos(\xi x)\, dx.$$

Also give a corresponding formula for odd functions.

74. Let $f \in L^2(\mathbb{R})$ be given by $f(t) = \sin(at)\sin(bt)/t$, with a and b positive. Show that f is band-limited in the sense that $\hat{f}(\omega) = 0$ for $|\omega| > a + b$.

75. If δ and Ω are two numbers with $0 < \pi/\Omega < \delta$, find a function $f \in L^2(\mathbb{R})$, such that $\hat{f}(\omega) = 0$ for $|\omega| \geq \Omega$ and $f(n\delta) = 0$ for $n \in \mathbb{Z}$, but $f \neq 0$ as an element of $L^2(\mathbb{R})$. This shows that the Shannon–Nyquist sampling distance $\Delta t = \pi/\omega_{\max}$ is the largest possible.
Hint: Use the previous Exercise.

76. Use Laplace transform to solve the initial value problems

    (a) $x''(t) - 3x'(t) + 2x(t) = e^t, \qquad x(0) = x'(0) = 0,$

    (b) $x''(t) - 2x'(t) + x(t) = e^t, \qquad x(0) = 0, \quad x'(0) = 1.$

77. Compute the Laplace transform of the function

$$f(x) = \begin{cases} x - x^2, & 0 \leq x \leq 1, \\ 0, & x > 1. \end{cases}$$

78. Find a function with Laplace transform $(1 - e^{-s})^2/s^2$. Sketch the graph of the function.

79. Let

$$f(t) = \begin{cases} \sin t, & 0 \leq t \leq \pi, \\ 0, & \text{else.} \end{cases}$$

Use Laplace transform to solve the initial value problem

$$x'(t) + x(t) = f(t), \qquad x(0) = 0.$$

80. Let

$$f(t) = \begin{cases} t, & 0 < t < 2, \\ 2, & t > 2. \end{cases}$$

Using Laplace transform, solve the initial value problem

$$x''(t) + x(t) = f(t), \qquad x(0) = 0, \quad x'(0) = 1.$$

81. Find a function with Laplace transform

$$\frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(1 + s^2)}.$$

Sketch the graph of the function.

82. Determine the finite (discrete) Fourier transform of the sequence $(1, 0, 0, -1)$. Check the inversion formula.

83. Give a version of Plancherel's formula for finite Fourier transform.

84. Let $x(n)$ be N-periodic, and

$$x(n) = \begin{cases} 1, & 0 \leq n \leq k-1, \\ 0, & k \leq n \leq N-1. \end{cases}$$

Compute the discrete Fourier transform. Using Parseval's formula, compute the sum

$$\sum_{\mu=1}^{N-1} \frac{1-\cos\frac{2\pi\mu k}{N}}{1-\cos\frac{2\pi\mu}{N}}.$$

85. Determine the discrete Fourier transform of the N-periodic function $x(n) = \sin\dfrac{n\pi}{N}, \quad n = 0, \ldots, N-1$.

86. Which of the following functions are piecewise continuous? Which are piecewise C^1?

(a) $f(t) = \begin{cases} \sin(1/t), & t > 0, \\ 0, & t \leq 0, \end{cases}$

(b) $f(t) = \begin{cases} t\sin(1/t), & t > 0, \\ 0, & t \leq 0, \end{cases}$

(c) $f(t) = \sqrt[3]{t}$,

(d) $f(t) = \begin{cases} t\{1/t\}, & t > 0, \\ 0, & t \leq 0, \end{cases}$

where $\{t\}$ denotes the fractional part, e.g. $\{\pi\} = 0.1415\ldots$.

Answers and hints:

1. (b) $x^2 + 4t^2 + xt$.

2. $u(x,t) = \sin(2x)\cos(2ct) + \dfrac{3}{5c}\sin(5x)\sin(5ct)$.

5. $\dfrac{e^{\cos(4\pi^2)} + e}{2}$.

6. $c_{2k} = (-1)^{k+1}/2k$, $c_{2k+1} = 2(-1)^k/\pi(2k+1)^2$. The value is $\pi/4$.

7. $12\sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n^3}$. 0 resp $\frac{3}{8}\pi^3$.

8. The Fourier series is
$$\frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{1+n^2}$$
and the sum is
$$\frac{\pi}{2}\coth(\pi) + \frac{1}{2}.$$

9. The Fourier series is
$$\frac{a\sin(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{inx}}{a^2-n^2}.$$

10. The integral equals $\pi/2$.

11. $\cos x = \dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{n}{4n^2-1}\sin 2nx \quad (0 < x < \pi)$. The value of the sum is $\pi^2/64$.

12. $u(x,t) = \dfrac{1}{2}(e^{-75t}\sin(5x) - e^{-27t}\sin(3x))$.

13. $u(x,t) = \dfrac{4}{\pi}\sum_{k=0}^{\infty}\dfrac{\sin((2k+1)x)e^{-2(2k+1)^2 t}}{2k+1}$.

14. $u(x,t) = \dfrac{8}{\pi}\sum_{k=1}^{\infty}\dfrac{k}{(2k+3)(2k-3)}\sin(2kx)e^{-8k^2 t}$.

15. $u(x,t) = \dfrac{1}{2} + \dfrac{2}{\pi}\sum_{k=0}^{\infty}\dfrac{(-1)^k}{2k+1}\cos((2k+1)x)e^{-(2k+1)^2 t}$.

16. $u(x,t) = \dfrac{2}{\pi} + \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{4n^2-1}\cos(nx)e^{-n^2 t}$.

17. (a) $T - \frac{4T}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\pi^2 at/l^2}\sin((2k+1)\pi x/l)$.

    (b) The cooking time is approximately $\frac{l^2}{a\pi^2}\ln\left(\frac{16}{3\pi}\right)$, so a twice as thick steak takes four times as long to cook.

18. Second question: In one of the points $a = kl/7$, $k = 1, 2, \dots, 6$. Note that these are the nodes of the sixth overtone.

19. Second question: Yes, take $\delta = kl/7$, $k = 1, 2, \dots, 6$.

20. $u(x,t) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n-1}}{n^2\pi^2}e^{-\sqrt{\frac{n\pi}{2}}}\cos\left(n\pi t - \sqrt{\frac{n\pi}{2}}x\right)$

21. $\frac{2}{3\ln 2}\ln r + \frac{2}{\pi^2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{(-1)^{n+1}}{n^2(2^n - 2^{-n})}(r^n - r^{-n})e^{in\theta}$

22. $u(x,y) = \frac{1}{6}(y^3 - y) + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\sinh 2n\pi}(\sinh n\pi x + 7\sinh n\pi(2-x))\sin n\pi y$

23. $u(x,t) = 1 - x + \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-\frac{2}{3}(2k+1)^2\pi^2[(1+t)^{3/2}-1]}\sin(2k+1)\pi x$

24. $$u(x,y) = -\frac{8}{\pi^3}\sin\pi x\frac{\sin(\sqrt{20-\pi^2}y)}{\sin\sqrt{20-\pi^2}} -$$
    $$-\frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k+1)^3}\sin(2k+1)\pi x\frac{\sinh(\sqrt{(2k+1)^2\pi^2-20}y)}{\sinh\sqrt{(2k+1)^2\pi^2-20}}$$

25. $u(x,t) = 1 + \frac{1}{2}(1 - e^{-2t})\cos x$.

26. $u(x,t) = e^{-t}\sin(x) + \frac{2\sin(\pi^2)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n^2-\pi^2)n^3}\left(e^{-n^2t} - 1 + n^2t\right)\sin(nx)$

27. $u(x,t) = \frac{x}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi n}e^{-n^2(t+t^2/2)}\sin(nx)$.

28.

$$\begin{aligned} u(x,t) &= (t+1)(1-x) + \sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}(e^{-2n^2\pi^2 t} - 1)\sin n\pi x \\ &= (t+1)(1-x) + \frac{x^2}{4} - \frac{x}{6} - \frac{x^3}{12} + \sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}e^{-2n^2\pi^2 t}\sin n\pi x \end{aligned}$$

29. $u(x,t) = -\frac{x^2}{2} + \frac{16}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\cos((2k+1)\pi t)\sin(\frac{(2k+1)\pi}{2}x)$

30. $u(x,y) = \frac{1}{2}(y^2 - x^2) + x + \frac{1}{2}$, or in polar coordinates $u(r,\theta) = -\frac{1}{2}r^2\cos 2\theta + r\cos\theta + \frac{1}{2}$.

31. False.

32. $u(x) = A\cos x + B\sin x$.

33. $x(t) = \dfrac{1}{2} - \displaystyle\sum_{n=1}^{\infty} \frac{\sin(2\pi nt - 2\arctan(2\pi n))}{\pi n(1+4\pi^2 n^2)}$.

34. $y = -\frac{4}{3} + \frac{2}{\pi^2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1}(1+in\pi)}{n^2(2n^2\pi^2+in\pi+1)}\, e^{in\pi x}$

35. $y(t) = \dfrac{2}{9} - \displaystyle\sum_{n=1}^{\infty} \frac{3(1-\cos\frac{2n\pi}{3})}{\pi^2 n^2(3-\frac{4}{9}n^2\pi^2)}\cos\frac{2n\pi t}{3}$

36. **Hint:** Write $\langle u,v\rangle - \langle u_n, v_n\rangle = \langle u, v - v_n\rangle + \langle u - u_n, v_n\rangle$.

37. **Hint:** Use the previous exercise.

38. For instance 1 and $\sqrt{3}(2x-1)$.

39. $a = 6(3-e)$, $b = 2(2e-5)$.

40. $\dfrac{2\sinh 1}{\frac{1}{2}\sinh 2 + 1}\cosh x + \dfrac{2e^{-1}}{\frac{1}{2}\sinh 2 - 1}\sinh x$

41. (a) $\dfrac{\sqrt{\pi}}{16}(3 + 6x - \dfrac{1}{2}x^2)$ (b) $\dfrac{8}{81}(x^2+12)$

42. $x^2 - x + \frac{1}{6}$

43. (c) $n!/\sqrt{2n+1}$.

44. $a = (e^{2\pi}-1)(1+i)/4\pi$, $b = (e^{2\pi}-1)(1-i)/4\pi$, minimum $(\pi(e^{4\pi}-1)-(e^{2\pi}-1)^2)/2\pi$.

45. $\pi^2/6$.

46. $\dfrac{\pi^4}{90}$.

47. $\pi^6/960$.

48. The Fourier expansion is $\cos^n x = \sum_{k=0}^{n}\binom{n}{k}e^{(2k-n)ix}$ and the value of the sum is $\binom{2n}{n}$.

49. 1, for any x.

50. $\dfrac{4}{\pi}\sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k+1)^2}\,e^{-(2k+1)^2t}\sin((2k+1)x)$.

51. Eigenvalues $\lambda_k=\nu_k^2$, where ν_k are the positive solutions to $\tan\nu a=\frac{3\nu}{2\nu^2-1}$.
Eigenfunctions: $\nu_k\cos\nu_k x+\sin\nu_k x$.

52. $\lambda_1=4-\beta_1^2$, where β_1 is the positive root of $\tanh\beta=\dfrac{\beta}{2}$; $u_1(x)=e^{-2x}\sinh\beta_1 x$
$\lambda_n=4+\beta_n^2$, where β_n, $n=2,3,\dots$ are the positive roots of $\tan\beta=\dfrac{\beta}{2}$; $u_n(x)=e^{-2x}\sin\beta_n x$
$$e^{-2x}=\sum_{n=1}^{\infty}\frac{2\sqrt{\lambda_n}[\sqrt{\lambda_n}+2(-1)^n]}{\beta_n(\lambda_n-2)}\,u_n(x)$$

54. (a) $2i\dfrac{\xi\cos\xi-\sin\xi}{\xi^2}$, (b) $\dfrac{2i\sin(\pi\xi)}{\xi^2-1}$, (c) $\dfrac{1}{1+i\xi}$, (d) $\dfrac{2(\xi^2+2)}{\xi^4+4}$, (e) $\dfrac{1}{2}\pi e^{3i\xi-2|\xi|}$, (f) $-\dfrac{1}{2}\pi i\xi e^{-|\xi|}$, (g) $\dfrac{\pi}{2}(1+|\omega|)e^{-|\omega|}$.

55. $\pi e^{-|a|}$.

56. $\dfrac{1}{4}\left(e^{\frac{1}{2}i(\xi-1)}\hat{f}\left(\dfrac{\xi-1}{2}\right)+e^{\frac{1}{2}i(\xi+1)}\hat{f}\left(\dfrac{\xi+1}{2}\right)\right)$.

58. $u(x)=(1+12x^2-16x^6)e^{-x^4}/2$.

59. $\pi^3/2a^2b^2(a+b)$.

60. (a) $11\pi/432$, (b) $3\pi/8$, (c) π/e, (d) $\pi\sin(1)$, (e) $\pi(1-e^{-1})$.

61. $\pi^2/2$.

62. a) i b) $\dfrac{i}{2\sqrt{2}}$

63. $\dfrac{1}{9\pi}$

64. $\hat{f}(\omega)=\dfrac{2\pi\sqrt{\omega}}{1+\omega}$ when $0<\omega<2$, 0 else. a) $\dfrac{\pi}{2}$, b) $2\pi\Big(\ln 3-\dfrac{2}{3}\Big)$

65. $\dfrac{\pi}{8}(e^2+1)$

66. $f(0)=1$, $\int_{-\infty}^{\infty}f(x)dx=1$.

68. $c_n=\begin{cases}\pi(e^{\frac{1}{2}}-e^{-\frac{1}{2}})e^{-|n|}, & n\neq 0\\ 2\pi(1-e^{-\frac{1}{2}}), & n=0.\end{cases}$
The system is not complete.

69. $c_n = 1 - \cos(n)/\pi n$. The minimum value is $1/8\pi$.

70. $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-1}^{1} e^{-(x-y)^2/4t}\, dy.$

71. $u(x,t) = \dfrac{4kt+1-2x^2}{(4kt+1)^{5/2}} e^{-\frac{x^2}{4kt+1}}$

72. $u(x,y) = \dfrac{1}{\pi}\left(\arctan\left(\dfrac{x+1}{y}\right) - \arctan\left(\dfrac{x-1}{y}\right)\right)$. Geometrically, $u(x,y) = \theta/\pi$, where θ is the angle at (x,y) in the triangle with corners (x,y), $(-1,0)$ and $(1,0)$.

73. See Folland, page 238.

76. (a) $x(t) = e^{2t} - (t+1)e^t$, (b) $x(t) = \left(\frac{t^2}{2} + t\right) e^t$.

77. $F(s) = \dfrac{s - 2 + e^{-s}(s+2)}{s^3}$.

78. $f(t) = \begin{cases} t, & 0 < t < 1, \\ 2-t, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$

79. $x(t) = \begin{cases} \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t, & 0 < t < \pi, \\ \frac{1}{2}(1+e^{\pi})e^{-t}, & t > \pi. \end{cases}$

80. $x(t) = \begin{cases} t, & t < 2, \\ 2 + \sin(t-2), & t > 2. \end{cases}$

81. $f(t) = |\sin t|$.

84. The value of the sum is $k(N-k)$.

85. $X(\mu) = \displaystyle\sum_{n=0}^{N-1} x(n) e^{-2\pi i \mu n/N} = \dfrac{\sin\frac{\pi}{N}}{\cos\frac{2\mu\pi}{N} - \cos\frac{\pi}{N}}$

86. (b) and (c) are piecewise continuous; none of the functions is piecewise C^1.