Supplementary exercises on Fourier series, Part 1. (Chapter 2.)

1. Find the Fourier series for the following 2π-periodic functions $f(x)$, both in sin/cosinus and complex forms (by by using elementary algebra instead of integration) and compte the integrals $\int_{-\pi}^{\pi} f(x)^2 dx$ and $\int_{-\pi}^{\pi} f(x) dx$.

$$a1)\, f(x) = \sin^2 x, \quad a2)\, f(x) = (\sin^2 x) \cos x, \quad a3)\, f(x) = \cos^3 x$$

(a bit more challengin)

$$a4)\, f(x) = \sin^n x, \quad a5)\, f(x) = \cos^n x$$

2. (Fourier analysis is very much related to complex analysis. The following exercises are simple examples of such relation.)

a) Find the sum of the following complex Fourier series.

$$\sum_{n=0}^{\infty} r^n e^{in\theta}, \quad 0 \le r < 1, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{1}{n} e^{in\theta}, \tag{2}$$

(Hint: Write $z = re^{i\theta}$ resp. $z = e^{i\theta}$ and compare with power series expansion of analytic functions.)

b) (More demanding question). Compute the real part of the above expansions to get Fourier sin/cos-series of real-valued functions.

3. Which of the following functions on $[-1, 1]$ are piece-wise continuous, piece-wise C^1? (Please see the textbook/lecture notes for the definition)?

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ -1, & x < 0 \end{cases}$$

$$g(x) = \begin{cases} x^2, & x \ge 0 \\ -1, & x < 0 \end{cases}$$

$$h(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n} \le x < \frac{1}{n-1}, \quad n = 2, 3, \cdots \\ 0, & x \le 0 \end{cases}$$