Supplementary Exercises on L^2-spaces and Sturm-Liouville problem. Part II (Chapt. 3)

1. Consider the space $\mathcal{P}_n$ of trigonometric polynomials of maximal degree n of period 2π, that is

$$\mathcal{P}_n = \text{Span}\{e^{ik\theta}; k = 0, \pm 1, \ldots, \pm n\}.$$

We equipp it with the Hermitian L^2-inner product

$$(f, g) = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}d\theta.$$

Let A, B, C, D be the following operator

$$A = \frac{d}{d\theta}, B = i\frac{d}{d\theta}, C = \frac{d^2}{d\theta^2},$$

and

$$D : f \mapsto Df(\theta) = f(\theta) + f(-\theta).$$

Which of the operators A, B, C, D are self-adjoint? Find the eigenvalues and eigenfunctions of A, B, C, D. (Non-self adjoint operators have compelx eigenvalues, generally.)

2. Which of the following series are point-wise convergent, absolutely convergent? Which ones are $L^2(-\pi, \pi)$-convergent?

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n+1} \cos(n\theta),$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{1}{n+1} (-1)^n \cos(n\theta),$$

$$(c) \quad \sum_{n=1}^{\infty} \frac{1}{n^2+1} \cos(n\theta)$$

$$(d) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos(n\theta)$$

$$(e) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \cos(n\theta)$$

$$(f) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \cos(n\theta)$$

$$(g) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \cos(n\theta)$$

(Hint: Use Abel's convergence criterion for the alternating series.)

3 [1] Find the eigenvalues and eigenfunctions for the following Sturm-Liouville problem on $[0, 1]$,

$$f'' + \lambda f = 0, \quad f'(0) = f(0), \quad f(1) = 0.$$

[1] Solution: $\lambda_n = \mu_n^2$, μ_n the solution of $\tan\mu = -\mu$; eigenfunction $f_n(x) = \mu_n \cos(\mu_n x) + \sin(\mu_n x)$