

## Supplementary Exercises on Fourier Transform (Chapter 7)

1. Let  $p \in \mathbb{R}$  and consider the function  $f(x) = |x|^p$ ,  $x \in \mathbb{R}$ .

(a) Prove that the function  $f$  is never in  $L^1(\mathbb{R})$  for any  $p$

(b) Consider now instead  $f$  as a function on the interval  $(0, \infty)$  (resp.  $(0, 1)$ ). Find conditions on  $p$  so that  $f \in L^1(0, \infty)$  (resp.  $L^1(0, 1)$ ).

2. Prove the formula for Fourier transform of  $f(x) = \sin x/x$  (see [F] Table 2. Entry 13) by completing the following steps.

(a). Solve the problem [E] 10, and prove that  $\int_{-\infty}^{\infty} f(x)dx = 2 \int_0^{\infty} f(x)dx = \pi$ . (This was done in an exercise session. Check your or your fellow students' notes)

(b) Prove  $\int_{-\infty}^{\infty} \frac{\sin(bx)}{x} dx = \text{sgn}(b)\pi$ . Here  $\text{sgn}(b)$  is the sign function, namely it's  $\pm$  according to  $b > 0$  or  $b < 0$ . (Hint. This follows immediately from (a) by a change of variables.)

(c) Compute the Fourier transform for  $f(x)$ . (Hint.  $f(x)$  is an even function so

$$\int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx = \int_{-\infty}^{\infty} f(x) \cos(-\xi x) dx = \int_{-\infty}^{\infty} f(x) \cos(\xi x) dx.$$

Now use  $\sin x \cos(\xi x) = \frac{1}{2}(\sin(\xi + 1)x - \sin(\xi - 1)x)$  and (b)).

3. Denote  $\chi_{[a,b]}(x)$  the characteristic function for the interval  $[a, b]$ , namely it is 1 on the interval and zero elsewhere. Consider the following functions

$$f(x) = \sum_{n=1}^{\infty} \chi_{[n, n + \frac{1}{n^2}]}(x),$$

$$g(x) = \sum_{n=1}^{\infty} \chi_{[n, n + \frac{1}{n}]}(x).$$

(a) Prove that  $f \in L^1(\mathbb{R})$ , and  $g \in L^2(\mathbb{R})$ , but  $g \notin L^1(\mathbb{R})$ .

(b). Prove that  $\hat{f}(\xi)$  is a continuous function, and that  $\hat{g}(\xi)$  is not continuous (actually it's not well-defined as point-wise function in the usual sense.)