EXAM, MMG 710 / TMA 362 Fourier A.

(Detailed) Solution (2016-10-22)

1. (a) The function f is continuous at $\theta = 0$

$f'(\theta)$ is PC^0. Then the F. S. of f converges

to $f(0)$ at $\theta = 0$:

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 0}{(2n-1)^2} = \frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{1}{(2n-1)^2},$$

$$\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(b) $F(\theta) - C\theta$ is 2π-periodic $\Leftrightarrow$

$$F(\theta+2\pi) - C(\theta+2\pi) = F(\theta) - C\theta$$

$$\Leftrightarrow \quad F(\theta+2\pi) - F(\theta) = 2\pi C$$

i.e. $$\int_{-\pi}^{\theta+2\pi} f(\varphi)\,d\varphi - \int_{-\pi}^{\theta} f(\varphi)\,d\varphi = 2\pi C$$

$$\Leftrightarrow \quad \int_\theta^{\theta+2\pi} f(\varphi)\,d\varphi = 2\pi C$$

But f is 2π-periodic, so this is

$$2\pi C = \int_0^{2\pi} f(\varphi)\,d\varphi, \qquad C = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi,$$

$$\int_{-\pi}^{\pi} f(\varphi)\,d\varphi = 2\int_0^{\pi} \varphi\,d\varphi = \pi^2. \qquad C = \frac{\pi}{2}.$$

[This can also be obtained immediately:

$$f(\theta)-\frac{\pi}{2} = -\frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2n-1)\theta}{(2n-1)^2}$$

the integral $\int_{-\pi}^{\pi}\left(f(\theta)-\frac{\pi}{2}\right)d\theta = 0 \Rightarrow$

$$\int_{-\pi}^{\theta}\left(f(\varphi)-\frac{\pi}{2}\right)d\varphi = F(\theta)-\frac{\pi}{2}\theta-\frac{\pi^2}{2} \text{ is periodic}$$

$\Rightarrow \quad c = \frac{\pi}{2}$]

Integrate $f(\varphi)-\frac{\pi}{2}$:

$$F(\theta)-\frac{\pi}{2}\theta = b+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\theta}{(2n-1)^3} \quad (*)$$

The series in convergent at $\theta = 0$:

$$F(0) = b+0, \qquad b = F(0) = \int_{-\pi}^{0}|\varphi|\,d\varphi = \frac{1}{2}\pi^2.$$

$$\Rightarrow \quad F(\theta)-\frac{\pi}{2}\theta = \frac{1}{2}\pi^2+\frac{4}{\pi}\sum_{1}^{\infty}\frac{\sin(2n-1)\theta}{(2n-1)^3}$$

[Alternatively b can be obtained by $\int_{-\pi}^{\pi}$ of $(*)$]

2 (a) We use Fourier cosine series.

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\Rightarrow \quad \operatorname{Span}\{1, \cos x\ \cos^2 x\} = \operatorname{Span}\{1, \cos x\ \cos 2x\}$$

Now $\cos^3 x = \cos x\,\frac{1}{2}(1+\cos 2x) = \frac{1}{2}\cos x+\frac{1}{2}\cos x\cos 2x$

and

$$\cos x \cos 2x = \frac{1}{2}(\cos 3x + \cos x)$$

$$\Rightarrow \quad \cos^3 x = \frac{1}{4}\cos 3x + p(x) = q(x) + p(x), \text{ where}$$

$$p(x) = \left(\frac{1}{2} + \frac{1}{4}\right)\cos x \quad \text{and} \quad p \in V.$$

But $\cos 3x \perp V$

$\Rightarrow$ The orthogonal projection of $\cos^3 x$ onto V is $p(x)$ and the distance from $f(x) = \cos^3 x$ to V is

$$\|q\| = \left\|\frac{1}{4}\cos 3x\right\| = \frac{1}{4}\cdot\sqrt{\pi}.$$

ANSWER: $\frac{1}{4}\sqrt{\pi}$.

(b) $\{1, \cos x, \cdots \cos^n x\}$ are linear combinations of $\{1, \cos x, \cdots \cos nx\}$ by repeatedly using the formula $\cos(\alpha+\beta) - \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$, and vice versa $\Rightarrow$

$$V = \operatorname{Span}\{1, \cos x, \cdots \cos nx\}$$

Now $\cos^{n+1} x = \frac{1}{2^{n+1}}(e^{ix} + e^{-ix})^{n+1}$ has its leading term $\frac{1}{2^{n+1}}\left(e^{i(n+1)x}+e^{-i(n+1)x}\right)$

$$= \frac{1}{2^{n+1}}\, 2\cos(n+1)x = \frac{1}{2^{n}}\cos(n+1)x$$

and the rest terms are in V, and $V \perp \cos(n+1)x$.

$\Rightarrow$ The distance from $\cos^{n+1}x$ to V is

$$\left\|\frac{1}{2^{n+1}}\cos(n+1)x\right\| = \frac{\sqrt{\pi}}{2^{n}}$$

3. $\mathcal{F}$: $e^{-\frac{1}{4}x^2} \longrightarrow 2\sqrt{\pi}\, e^{-\xi^2}$

$$e^{-\frac{1}{4}(x-1)^2} \longrightarrow 2\sqrt{\pi}\, e^{-\xi^2} e^{-i\xi}$$

$$e^{\pm i2x} e^{-\frac{1}{2}(x-1)^2} \longrightarrow 2\sqrt{\pi}\, e^{-(\xi\mp 2)^2} e^{-i(\xi\mp 2)}$$

$\Rightarrow$ $\sin 2x\, e^{-\frac{1}{2}(x-1)^2} \longrightarrow \frac{1}{2i}\, 2\sqrt{\pi}\left(e^{-(\xi-2)^2}e^{-i(\xi-2)} - e^{-(\xi+2)^2}e^{-i(\xi+2)}\right)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \hat{f}(0) = \frac{2\sqrt{\pi}}{2i}\left(e^{-4}e^{2i} - e^{-4}e^{-2i}\right)$$

$$= \underline{2\sqrt{\pi}\, e^{-4} \sin 2}$$

$$\int_{-\infty}^{\infty} x f(x)\,dx = i(\hat{f})'(0) = i\,\frac{2\sqrt{\pi}}{2i}\Big(\left(-2(-2)-i\right)e^{-4}e^{2i}$$
$$-\left(-2(2)-i\right)e^{-4}e^{-2i}\Big)$$

$$= \sqrt{\pi}\, e^{-4}\left((4-i)e^{2i} - (-4-i)e^{-2i}\right)$$

$$= \sqrt{\pi}\, e^{-4}\left(8\cos 2 + 2\sin 2\right)$$

$$= \underline{2\sqrt{\pi}\, e^{-4}\left(4\cos 2 + \sin 2\right)}.$$

4. The general solution is of the form

$$u(x,t) = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})^2 t}\, b_n \sin\left(n+\tfrac{1}{2}\right)x,$$

by sep. of variables and solving the S-L-problem.

$\boxed{t=0}$

$$u(x, 0) = \frac{1}{2}\left(\sin\left(\tfrac{3}{2}x\right) - \sin\left(\tfrac{1}{2}x\right)\right).$$

$\Rightarrow$ $b_n = 0$ if $n \neq 0, 1$ and

$$b_0 = -\frac{1}{2}, \quad b_1 = \frac{1}{2}$$

$$\Rightarrow \quad u(x,t) = -\frac{1}{2} e^{-(\frac{1}{2})^2 t} \sin\frac{1}{2}x + \frac{1}{2} e^{-(\frac{3}{2})^2 t} \sin\frac{3}{2}x$$

5. Write $U(z) = \mathcal{L}[u](z)$.

$$(z^2 - 2z)\,U(z) = \frac{1}{z} - e^{-z}\frac{1}{z} = \frac{1}{z}\left(1 - e^{-z}\right)$$

$$U(z) = \frac{1-e^{-z}}{z^2(z-2)} = \left(\frac{A}{z^2} + \frac{B}{z} + \frac{C}{z-2}\right)\left(1-e^{-z}\right).$$

$$z^2\frac{1}{z^2(z-2)} = \left(A + Bz + \frac{C}{z-2}z^2\right). \quad \text{Let } z \to 0$$

$-\frac{1}{2} = A$. $\frac{d}{dz}$ and let $z \to 0$: $B = \frac{1}{4}$ (6)

Similarly $C = \frac{1}{4}$ and

$$\frac{1}{z^2(z-2)} = -\frac{1}{2}\frac{1}{z^2} + \frac{1}{4}\frac{1}{z} + \frac{1}{4}\frac{1}{z-2}$$

$$U(z) = \left(-\frac{1}{2}\frac{1}{z^2} + \frac{1}{4}\frac{1}{z} + \frac{1}{4}\frac{1}{z-2}\right)(1-e^{-z})$$

$$u(t) = \mathcal{L}^{-1}[U](t)$$

$$= \left(-\frac{1}{2}t^2 + \frac{1}{4} + \frac{1}{4}e^{2t}\right)H(t)$$

$$-\left(-\frac{1}{2}(t-1)^2 + \frac{1}{4} + \frac{1}{4}e^{2(t-1)}\right)H(t-1).$$

$u(t)$ is not supported on $[0, 1]$ since

for $t > 1$, $u(t) = \frac{1}{2}(t-1)^2 - \frac{1}{2}t^2 + \frac{1}{4}(e^{2t} - e^{2(t-1)})$

is not zero, generally.

6. See the text book.