Mathematical Sciences (MV), Chalmers/GU
Saturday, October 21, 2017, 8:30-12:30.

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Presence of teacher: $\sim 9:30$ and $\sim 11:30$

Tools: Only the attached sheets (2 pages) of formulas. No calculator or handbook is allowed.
(Language Dictionaries are allowed/Språlektion är tillåtet)

Exam in MMG710/TMA362 Fourier Analysis

1. Let f be 2π-periodic and $f(x) = x$, $-\pi < x < \pi$. The Fourier series of f is given by

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nx}{n} \tag{1}$$

(**a**) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(**b**) Find the sum of the Fourier series $\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$, and $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, for $0 < x < \pi$.

2. Let $n \geq 1$ be a fixed integer and let V be the $2n+1$-dimensional subspace $V = \text{Span}\{e^{ikx}\}_{k=-n}^{n}$ of the Hilbert space $L^2(-\pi, \pi)$ spanned by the vectors $\{e^{ikx}\}_{k=-n}^{n}$. Let $f(x) = (\cos x)^{n+2}$ and $g(x) = (\cos nx)\sin x$.

(**a**) Find the distance between g and V.

(**b**) Find the orthogonal projection of f onto the subspace V.

(Hint: Use Euler's formula and binomial expansion.)

3. Let $f(x) = \frac{\cos x}{x^2+1}$, $g(x) = \frac{\sin x}{x}$, and $h(x) = f * g(x)$.

(**a**) Find the Fourier transform $\widehat{f}(\xi)$.

(**b**) Prove that h is band-limited, i.e., $\hat{h}(\xi) = 0$ outside an interval, and compute and the integral $\int_{-\infty}^{\infty} h(x)^2 dx$.

4. Solve the following Dirichlet problem on the square $[0, \pi]^2$:

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = 0, & 0 \leq x, y \leq \pi \\ u(0, y) = u(\pi, y) = 0, & 0 < y < \pi \\ u(x, 0) = f(x),\ u(x, \pi) = 0, & 0 < x < \pi \end{cases}$$

where $f(x)$ is the function in the Problem 1.

5. Solve the following ODE

$$u'' + 3u' + 2u = H(t-1)e^{c(t-1)},\ u(0) = 0,\ u'(0) = 0,$$

where $c \in \mathbb{R}$ is a constant, $c \neq -1, c \neq -2$. For which c is your solution unstable, i.e. $\lim_{t\to\infty} u(t) = \infty$?

6. Let f be 2π-periodic and $f \in C^0 \cap PC^1$, i.e. f is continuous and piece-wise differentiable. Prove that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. Provide an example of a convergent Fourier series which is not uniformly convergent.

Grades: 6 problems each of 4 points.
MMG710: G (12-17 p.), VG (18-24 p.). **TMA362**: 3 (12-14 p.), 4 (15-17 p.), 5 (18-24 p.)

Basic Fourier Transforms and Laplace Transforms

0.1 Fourier transform $\mathcal{F} : f(x) \mapsto \widehat{f}(\xi)$. ($a > 0$ and $c \in \mathbb{R}$ are constants)

$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
$e^{i\alpha x}f(x)$	$\widehat{f}(\xi-\alpha)$
$f(ax)$	$a^{-1}\widehat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\widehat{f}(\xi)$
$xf(x)$	$i(\widehat{f})'(\xi)$
$(f*g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f}*\widehat{g})(\xi)$
$e^{-a\,\frac{x^2}{2}}$	$\sqrt{\frac{2\pi}{a}}\,e^{-\frac{\xi^2}{2a}}$
$(x^2+a^2)^{-1}$	$\frac{\pi}{a}e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x)$	$2\xi^{-1}\sin a\xi$
$x^{-1}\sin ax$	$\pi\chi_a(\xi)$

0.2 Laplace transforms $\mathcal{L} : f(t) \mapsto F(z) = \mathcal{L}f(z)$. ($a > 0$ and $c \in \mathbb{C}$ are constants)

$H(t-a)f(t-a)$	$e^{-az}F(z)$
$e^{ct}f(t)$	$F(z-c)$
$f(at)$	$a^{-1}F(a^{-1}z)$
$f'(t)$	$zF(z)-f(0)$
$f''(t)$	$z^2F(z)-zf(0)-f'(0)$
$f*g$	FG
$H*f(t)=\int_0^t f(s)ds$	$z^{-1}F(z)$
$tf(t)$	$-F'(z)$
$t^n e^{ct}$	$\frac{n!}{(z-c)^{n+1}}$
$\sin ct$ resp. $\cos ct$	$\frac{c}{z^2+c^2}$ resp. $\frac{z}{z^2+c^2}$
$\sinh ct$ resp. $\cosh ct$	$\frac{c}{z^2-c^2}$ resp. $\frac{z}{z^2-c^2}$

Some formulas in Fourier analysis

Trigonometric identities

$$e^{ix} = \cos x + i \sin x, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \qquad \sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin x \ \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}, \qquad \cos x \ \cos y = \frac{\cos(x-y) + \cos(x+y)}{2},$$

$$\sin x \ \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}.$$

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Characteristic and Heaviside functions

$$\chi_{[a,b]}(x) = \begin{cases} 1, & a < x < b, \\ 0, & \text{otherwise.} \end{cases} \qquad \chi_a(x) := \chi_{[-a,a]}(x).$$

$$H(t) = \chi_{[0,\infty)}(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{else.} \end{cases}$$

Gamma function $\Gamma(a)$

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx, \qquad a > 0$$

$$\Gamma(a+1) = a\Gamma(a), \qquad \Gamma(n+1) = n!$$