Solution (with details). Exam in MMG710/TMA362 Fourier Analysis, 2014-10-27

1. In which space C^k are the following periodic functions? Find the best (i.e. the largest) k.

$$(\mathbf{a}) \quad \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2}, \; (\mathbf{b}) \quad \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{5^n}, \; (\mathbf{c}) \quad \sum_{n=1}^{\infty} \frac{\sin(2^n\theta)}{5^n}.$$

Motivate your answers.

<u>Solution</u> We use the Theorem on term-wise differentiation of Fourier series. Write the given series as $f(\theta)$. In the cases (a)-(b) $f(\theta)$ is a well-defined convergent series and thus a well-defined function.

(a) Differentiating the series formally term-wise we get a series

$$\sum_{n=1}^{\infty} \frac{d}{d\theta} \frac{\cos(n\theta)}{n^2} = -\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$$

which is convergent but not absolutely convergent. Thus (the best we can say) f is C (but not C^1).

(b). The series $\sum_{n=-\infty}^{0} \frac{(in)^j e^{in\theta}}{5^n}$ is divergent. Thus the series has no regularity. (However the series $\sum_{n=1}^{\infty} \frac{(in)^j e^{in\theta}}{5^n}$ is C^∞, since diferentiating term-wise of the series j-times results in an absolutely convergent series, for $\sum_{n=1}^{\infty} \frac{n^j}{5^n} < \infty$.)

(c) We perform differentiation twice on the series, and find

$$-\sum_{n=1}^{\infty} \frac{(2^n)^2 \sin(2^n\theta)}{5^n},$$

which is absolutely dominated by $\sum_{n=1}^{\infty} \frac{(2^n)^2}{5^n} = \sum_{n=1}^{\infty} (\frac{4}{5})^n < \infty$, whereas differentiating one more time it is

$$-\sum_{n=1}^{\infty} \frac{2^{3n} \sin(2^n\theta)}{5^n},$$

which is divergent, e.g. for $\theta = \frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{7}$ etc. Thus $f \in C^2$.

2. Compute the following integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)\cos(2x)}{x(x^2+1)} dx$$

<u>Solution</u> Write $\sin(x)\cos(2x) = \frac{1}{2}(\sin(3x) - \sin(x))$ and thus

$$I := \int_{-\infty}^{\infty} \frac{\sin(x)\cos(2x)}{x(x^2+1)} dx = \frac{1}{2}(\int_{-\infty}^{\infty} \frac{\sin(3x)}{x(x^2+1)} dx - \int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+1)} dx)$$

We compute, for any $a > 0$, the integral $\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)} dx$. Write $f(x) = \frac{\sin(ax)}{x}, g(x) = (x^2+1)^{-1}$ and use Plancherel formula:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)} dx &= (f, g) = \frac{1}{2\pi}(\hat{f}, \hat{g}) \\ &= \frac{1}{2\pi}(\pi\chi_a, \pi e^{-|\cdot|}) = \frac{\pi}{2}\int_{-a}^{a} e^{-|\xi|} d\xi \\ &= \pi \int_0^a e^{-\xi} d\xi = \pi(1 - e^{-a})\end{aligned}$$

Thus taking $a = 3, 1$ we find

$$I = \frac{1}{2}(\pi(1 - e^{-3}) - \pi(1 - e^{-1})) = \frac{\pi}{2}(e^{-1} - e^{-3}).$$

3. Solve the following ordinary differential equation

$$u''(t) - 4u(t) = f(t),\ u(0) = 0,\ u'(0) = 1,$$

where

$$f(t) = H(t-1) = \begin{cases} 1, & t \geq 1 \\ 0, & \text{else} \end{cases}$$

Solution We apply the Laplace transform $\mathcal{L}$ to the equation, writing $\mathcal{L}u(z) = U(z)$,

$$z^2U(z) - u'(0) - zu(0) - 4U(z) = 8\mathcal{L}[H(t-1)](z) = 8\frac{e^{-z}}{z}.$$

$$(z^2 - 4)U(z) - 1 = 8\frac{e^{-z}}{z}.$$

Solve $U(z)$ and perform partial fractional decompositions:

$$U(z) = 8e^{-z}\frac{1}{z(z-2)(z+2)} + \frac{1}{z^2 - 2^2} = e^{-z}(\frac{-2}{z} + \frac{1}{z-2} + \frac{1}{z+2}) + \frac{1}{z^2 - 2^2}$$

Its inverse transform gives the solution

$$u(t) = H(t-1)(-2 + e^{2(t-1)} + e^{-2(t-1)}) + \frac{1}{2}\sinh(2t) = 2H(t-1)\left(-1 + \cosh 2(t-1)\right) + \frac{1}{2}\sinh(2t)$$

4. Solve the following inhomogeneous wave equation

$$\begin{cases} u_{tt} = c^2u_{xx} + t\sin(2x), & t > 0, \quad 0 < x < \pi \\ u(0,t) = 0,\ u(\pi,t) = 0, & t > 0 \\ u(x,0) = x(\pi - x), & 0 < x < \pi \end{cases}$$

You may use (without proof) the following Fourier sine series on $(-\pi, \pi)$

$$\theta(\pi - |\theta|) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\theta}{(2n-1)^3}.$$

Solution . We find first a special solution solving the inhomogeneous equation and preserving the homogeneous condition. Ansats

$$\boxed{w(x,t) = \frac{1}{c^2}t\sin(2x)}.$$

Then $w_{tt} = 0$ and $c^2w_{xx} = -t\sin(2x)$, namely $c^2w_{xx} + t\sin(2x) = 0$, w solves indeed the inhomogeneous equation along with the boundary condition since $\sin(2x) = 0$ for $x = 0, \pi$. Now writing $u = w + v$ and $u_t(x,0) = g(x)$ the function v satisfies

$$\begin{cases} v_{tt} = c^2v_{xx} \\ v(0,t) = 0,\ v(\pi,t) = 0 \\ v(x,0) = x(\pi - x), \quad v_t(x,0) = g(x) - \frac{2}{c^2}\sin(2x), \end{cases}$$

Let β_n be the Fourier sine coefficients of g and write

$$\boxed{\alpha_n = \begin{cases} \beta_n, & n \neq 2 \\ \beta_2 - \frac{2}{c^2}, & n = 2. \end{cases}}$$

The Fourier sine series of $v(x,0)=x(\pi-x)$ on $(0,\pi)$ is given by $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$, whereas the Fourier sine series of $v_t(x,0)$ is $\sum_{n=1}^{\infty}\alpha_n \sin nx$. The solution for v is given by

$$\boxed{v(x,t)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)ct)\sin((2n-1)x)}{(2n-1)^3}+\sum_{n=1}^{\infty}\frac{1}{nc}\alpha_n\sin(nct)\sin nx}$$

Answer: $\boxed{u(x,t)=w(x,t)+v(x,t)}$.

5. Evaluate the sum of the following series by using the above Fourier expansion. Motivate your answer.

$$(\mathbf{a})\quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},\qquad (\mathbf{b})\quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}.$$

<u>Solution</u> The function $f(x)=x(\pi-|x|)$ on $(-\pi,\pi)$ has continuous derivative and piece-wise second order derivative: $f'(x)=\pi-2x\text{sgn}(x)$ is continuous, $f''(x)=2\text{sgn}(x)$ is piece-wise continuous. Thus $f'(x)$ has its Fourier series given by term-wise differentiation and the series converges to $f'(x)$,

$$f'(x)=\pi-2x\text{sgn}(x)=\theta(\pi-|\theta|)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Taking $x=0$ we find

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\pi,\qquad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\pi^2/8,$$

We apply Parseval's (Pythagoras') theorem to the Fourier expansion of $f(x)$ on $(-\pi,\pi)$

$$\int_{-\pi}^{\pi}f(x)^2dx=\pi(\frac{8}{\pi})^2S,\quad S:=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6},$$

since $\int_{-\pi}^{\pi}\sin^2(2n-1)xdx=\pi$. Now the left hand side is

$$\int_{-\pi}^{\pi}f(x)^2dx=2\int_{-0}^{\pi}x^2(\pi-x)^2dx=2\int_{-0}^{\pi}(\pi^2x^2-2\pi x^3+x^4)dx=2(\pi^2\frac{\pi^3}{3}-2\pi\frac{\pi^4}{4}++\frac{\pi^5}{5})=\frac{\pi^5}{15}.$$

Thus

$$S=\frac{\pi^6}{2^6\cdot 15}.$$

6. Formulate and prove *the Theorem on Uniform Convergence for Fourier Series* of 2π-periodic C^1-functions.

<u>Solution</u> See the textbook.