Solution. Exam 15-08-17 MMG 710/TMA 362 (1

1. $f(x)$ is piece-wise differentiable, thus by term-wise differentiation

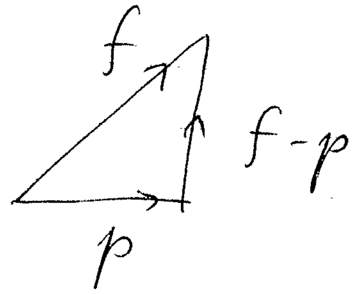
$$f'(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)\sin(2n-1)x}{2n-1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}.$$

Note that the integral $\int_{-\pi}^{\pi}\left(f(t)-\frac{\pi}{2}\right)dt = 0$, so the function $F(x) = \int_{-\pi}^{x}\left(f(t)-\frac{\pi}{2}\right)dt$ is 2π-periodic and its F. S is obtained by term-wise integration

$$F(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$$

(by using $\sin(2n-1)\pi = 0$)

2. We find the orthogonal projection of f and use Pythagoras theorem.



Set $e_0 = 1$, $e_1 = x + c$, to be an O-basis of V.

$$0 = \langle e_0, e_1\rangle = \frac{1}{2} + c, \Rightarrow c = -\frac{1}{2}.$$

O-proj of f on V is $p = \frac{\langle f, e_0\rangle}{\langle e_0, e_0\rangle}e_0 + \frac{\langle f, e_1\rangle}{\langle e_1, e_1\rangle}e_1$.

This is found by direct integration

$$p = \frac{1}{3}e_0 + e_1$$

Now use P- theorem:

$$\|f-p\|^2 = \|f\|^2 - \|p\|^2,$$

$$\|f\|^2 = \frac{1}{5}, \quad \|p\|^2 = \frac{1}{9} + \frac{1}{3\cdot 2^2} = \frac{4+3}{3^2\cdot 2^2} = \frac{7}{2^2\cdot 3^2}$$

$$\|f-p\|^2 = \frac{1}{5} - \frac{7}{2^2\cdot 3^2} = \frac{1}{2^2\cdot 3^2\cdot 5}.$$

Answer: $\dfrac{\sqrt{5}}{2\cdot 3\cdot 5}$.

3. (a) $f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i0\cdot\xi} d\xi = \frac{1}{2\pi}\int_0^1 \frac{1}{1+\xi} d\xi = \frac{1}{2\pi}\ln 2.$

$$f'(0) = \frac{1}{2\pi}\int_0^1 \frac{-i\xi}{1+\xi} d\xi = -\frac{i}{2\pi}\left[1-\ln 2\right]$$

(b) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \frac{1}{2\pi}\int_0^1 \frac{d\xi}{(1+\xi)^2} = \frac{1}{4\pi}$

5. (a) [The "continuous functions" should be "piece-wise continuous function] The F.S of $f(\theta)=\theta$, $-\pi<\theta<\pi$ has its F.S being $2\sum \frac{(-1)^{n+1}}{n}\sin n\theta$, which is not absolutely convergent. [The picture of f is wrong in Folland's book]

(b) See the text book.

6. See the text book

4. Put $V_0(x) = \frac{1}{c^2}\cos x$. Then $V_0(x)$ solves the non-homogeneous wave equation and the boundary condition. Let $u = V(x,t) + V_0(x)$. Then $v(x,t)$ should solve

$$\begin{cases} V_{tt} = c^2 V_{xx} & t>0, \quad x \in (0,\pi) \\ V_x(0,t) = 0, \quad V_x(\pi,t) = 0, & t>0 \\ V(x,0) = |x| - \frac{1}{c^2}\cos x, \quad V_t(x,0) = 0. \end{cases}$$

The general solution of this homogeneous eq. is

$$V(x,t) = \sum_{n=0}^{\infty} \cos(nx)\left(a_n \sin nct + b_n \cos nct\right).$$

$$0 = V_t(x,0) = \sum_{n=0}^{\infty} \cos(nx)\, a_n (nc) \quad \Rightarrow \quad a_n = 0 \ \forall n.$$

$$|x| - \frac{1}{c^2}\cos x = V(x,0) = \sum_{n=0}^{\infty} \cos(nx)\, b_n.$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}. \quad \Rightarrow \quad b_0 = \frac{\pi}{2} \quad b_1 = -\frac{4}{\pi} - \frac{1}{c^2},$$

$$b_{2m} = 0, \quad m \geq 1 \qquad b_{2m-1} = -\frac{4}{\pi}\frac{1}{(2m-1)^2}, \quad m>1.$$

Answer: $u = \frac{1}{c^2}\cos x + \frac{\pi}{2} + \left(-\frac{4}{\pi} - \frac{1}{c^2}\right)\cos x \cos ct$
$$+ \sum_{m>1} -\frac{4}{\pi}\frac{1}{(2m-1)^2}\cos(2m-1)x \cos(2m-1)ct.$$