

Tools: Only the attached sheets (2 pages) of formulas. No calculator or handbook is allowed.
(Language Dictionaries are allowed/Språkleksion är tillåtet)

Exam in MMG710/TMA362 Fourier Analysis

1. Let $f(x)$ be the 2π -periodic function on \mathbb{R} such that $f(x) = \pi - 2|x|$, $-\pi < x \leq \pi$. The Fourier expansion of $f(x)$ is

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}. \quad (1)$$

- (a) Find the Fourier series expansions of $f'(x)$ and $f''(x)$, both being viewed as functions in $L^2(-\pi, \pi)$. (b) Is the Fourier series of $f'(x)$ uniformly convergent? Motivate your answer.
2. Find the best approximation in the space $L^2(0, \infty)$ for the function e^{-2x} among all functions of the form $ae^{-x} + bxe^{-x}$, where $a, b \in \mathbb{R}$ are constants.

3. Let

$$f(x) = \frac{\sin^2 x}{x^2}.$$

- (a) Compute the integral $\int_{-\infty}^{\infty} f(x) dx$. (b) Prove that the Fourier transform $\widehat{f}(\xi)$ is supported on $[-2, 2]$, i.e. $\widehat{f}(\xi) = 0$ for $|\xi| > 2$.
4. Solve the following inhomogeneous heat equation

$$\begin{cases} u_t = k u_{xx} + \sin x, & t > 0, \quad 0 < x < \pi \\ u(0, t) = 0, \quad u(\pi, t) = 0, & t > 0 \\ u(x, 0) = f(x), & 0 < x < \pi, \end{cases}$$

where the Fourier series of the initial temperature $f(x)$ is given (abstractly) by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

5. Evaluate the following sums by using the Fourier expansion (1) above. Motivate your answer.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3}, \quad (b) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

6. Formulate and prove a version of *the Bessel inequality*. (Any version is fine.)

Grades: 6 problems each of 4 points.

MMG710: G (12-17 p.), VG (18-24 p.). **TMA362:** 3 (12-14 p.), 4 (15-17 p.), 5 (18-24 p.)

Basic Fourier Transforms and Laplace Transforms

0.1 Fourier transform $\mathcal{F} : f(x) \mapsto \widehat{f}(\xi)$. ($a > 0$ and $c \in \mathbb{R}$ are constants)

$f(x - c)$	$e^{-ic\xi} \widehat{f}(\xi)$
$e^{i\alpha x} f(x)$	$\widehat{f}(\xi - \alpha)$
$f(ax)$	$a^{-1} \widehat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi \widehat{f}(\xi)$
$xf(x)$	$i(\widehat{f})'(\xi)$
$(f * g)(x)$	$\widehat{f}(\xi)\widehat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\widehat{f} * \widehat{g})(\xi)$
$e^{-a\frac{x^2}{2}}$	$\sqrt{\frac{2\pi}{a}} e^{-\frac{\xi^2}{2a}}$
$(x^2 + a^2)^{-1}$	$\frac{\pi}{a} e^{-a \xi }$
$e^{-a x }$	$2a(\xi^2 + a^2)^{-1}$
$\chi_a(x)$	$2\xi^{-1} \sin a\xi$
$x^{-1} \sin ax$	$\pi\chi_a(\xi)$

0.2 Laplace transforms $\mathcal{L} : f(t) \mapsto F(z) = \mathcal{L}f(z)$. ($a > 0$ and $c \in \mathbb{C}$ are constants)

$H(t - a)f(t - a)$	$e^{-az}F(z)$
$e^{ct}f(t)$	$F(z - c)$
$f(at)$	$a^{-1}F(a^{-1}z)$
$f'(t)$	$zF(z) - f(0)$
$f''(t)$	$z^2F(z) - zf(0) - f'(0)$
$f * g$	FG
$H * f(t) = \int_0^t f(s)ds$	$z^{-1}F(z)$
$tf(t)$	$-F'(z)$
$t^n e^{ct}$	$\frac{n!}{(z-c)^{n+1}}$
$\sin ct$ resp. $\cos ct$	$\frac{c}{z^2+c^2}$ resp. $\frac{z}{z^2+c^2}$
$\sinh ct$ resp. $\cosh ct$	$\frac{c}{z^2-c^2}$ resp. $\frac{z}{z^2-c^2}$

Some formulas in Fourier analysis

Trigonometric identities

$$\begin{aligned}
 e^{ix} &= \cos x + i \sin x, & \cos x &= \frac{e^{ix} + e^{-ix}}{2}, & \sin x &= \frac{e^{ix} - e^{-ix}}{2i}, \\
 \cos(x + y) &= \cos x \cos y - \sin x \sin y, & \sin(x + y) &= \sin x \cos y + \cos x \sin y, \\
 \sin^2 x &= \frac{1 - \cos 2x}{2}, & \cos^2 x &= \frac{1 + \cos 2x}{2}, \\
 \sin x \sin y &= \frac{\cos(x - y) - \cos(x + y)}{2}, & \cos x \cos y &= \frac{\cos(x - y) + \cos(x + y)}{2}, \\
 \sin x \cos y &= \frac{\sin(x + y) + \sin(x - y)}{2}.
 \end{aligned}$$

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Characteristic and Heaviside functions

$$\chi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a. \end{cases}$$

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{else.} \end{cases}$$

Gamma function $\Gamma(a)$

$$\Gamma(a) = \int_0^{\infty} e^{-x} x^{a-1} dx, \quad a > 0$$

$$\Gamma(a + 1) = a\Gamma(a), \quad \Gamma(n + 1) = n!$$