Repetition Questions [1]

1. Formulate the definitions of the following concepts

   - Piece-wise continuous PC^0 functions, Piece-wise smooth PC^1 functions and generally PC^k functions.
   - Fourier series, Fourier transform, Laplace transform.
   - Self-adjoint Sturm-Liouville operator.
   - Convolution.
   - L^2-convergence.

2. Formulate and prove the following theorems

   - Bessel inequality.
   - Riemann-Lebesgue lemma for Fourier series and for Fourier transform.
   - Differentiation and integration of Fourier series. Fourier transform of differentiation.
   - Uniform convergence of Fourier series.
   - Best approximation of a given function in a L^2-space by linear combination of orthogonal functions.
   - Plancherel formula for F. transform.
   - Laplace inversion (in terms of Fourier transform) for $\mathcal{E} \cap C^0$-functions.

3. Formulate the following theorems and present some key steps of their proofs.

   - Point-wise convergence of Fourier series.
   - Parseval-Plancherel formula for F. series.
   - Fourier inversion formula (two versions)

4. Compute the following sums (by using Fourier series in Table 1)

$$\sum_{n=1}^{\infty} \frac{(\sin na)^2}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

5. Evaluate the following integrals using Fourier transform

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2} d\xi$$

6. Which of the following argument is valid/invalid, and why?

   (a) By differentiating the series

$$\theta = 2 \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta, \quad \theta \in (-\pi, \pi)$$

[1] These questions are intended only to help summarize some of the main concepts/theorems/tools. You may try more of the recommended exercises in [F][E].

we get

$$\frac{d}{d\theta}\theta = 1 = 2\sum_1^\infty \frac{(-1)^{n+1}}{n}\frac{d}{d\theta}\sin n\theta = 2\sum_1^\infty (-1)^{n+1}\cos n\theta \quad \theta \in (-\pi, \pi)$$

(b) Performing the differentiation on the series

$$|\sin\theta| = [\text{Table 1, Entry 8}] = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2 - 1}$$

we have

$$\frac{d}{d\theta}|\sin\theta| = (\cos\theta)\text{sgn}(\sin\theta) = \frac{4}{\pi}\sum_1^\infty \frac{2n\sin 2n\theta}{4n^2-1}$$

(c) We apply the general rule for Fourier transform for the differentation for the function $\chi_a(x)$ and find

$$\mathcal{F} : \chi_a(x) \mapsto 2\frac{\sin a\xi}{\xi}$$

$$\mathcal{F} : \frac{d}{dx}\chi_a(x) = 0 \mapsto i\xi 2\frac{\sin a\xi}{\xi} = 2i\sin a\xi$$

7. How smooth are the following functions f? Find best k (that is the largest k) so that the functions are in C^k.

$$(a) \qquad \sum_1^\infty \frac{1}{\sqrt{n}}\sin n\theta,$$

$$(b) \qquad \sum_1^\infty \frac{1}{n}\sin n\theta,$$

$$(c) \qquad \sum_1^\infty \frac{1}{n^{5/2}}\sin n\theta,$$

$$(d) \qquad \sum_1^\infty \frac{1}{2^n}\sin(n\theta),$$

8. Solve the followng homogeneous resp. inhomogeneous heat eq.

$$(a) \quad \begin{cases} u_t = u_{xx} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \sin\frac{\pi x}{l}\cos\frac{2\pi x}{l} \end{cases}$$

$$(b) \quad \begin{cases} u_t = u_{xx} + 1 \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \sin\frac{\pi x}{l}\cos\frac{2\pi x}{l} \end{cases}$$

9. Solve the following wave eq.

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(l,t) = 1 \\ u(x,0) = x, \quad u_t(x,0) = 0. \end{cases}$$

10. Solve the following Laplace eq's on the square $[0,l]^2$ and respectively on the unit disk:

$$(a) \begin{cases} \nabla^2 u(x,y) = 0, \quad (x,y) \in (0,l)^2 \\ u(x,0) = x, \quad u(l,y) = 0, \quad u(x,l) = x(l-x), \quad u(0,y) = 0 \end{cases}$$

$$(b) \begin{cases} \nabla^2 u(x,y) = 0, \quad x^2+y^2 < 1 \\ u|_{r=1} = \theta, ((r,\theta) \text{ are the polar coordinates} \end{cases}$$

$$(c) \begin{cases} \nabla^2 u(x,y) = 0, \quad x^2+y^2 < 1 \\ u|_{r=1} = \sin\theta\cos^2\theta, ((r,\theta) \text{ are the polar coordinates} \end{cases}$$

(This problem can be solved using trigonometry, writing $\sin\theta\cos^2\theta$ as a sum of $e^{\pm in\theta}, n = 1,2,3$.)

11. Solve the following ODEs (by using Laplace transform)

$$u'' + u' + 12u = e^t(H(t) - H(t-1)), u(0) = 1, u'(0) = 1.$$

12. Find the best approximation of the function x^2e^{-x} by functions of the form $(a+bx)e^{-x}$ in the space $L^2(0,\infty)$, i.e. find the costants a, b which minimize the L^2-norm square

$$\int_0^\infty |(a+bx)e^{-x} - x^2e^{-x}|^2 dx$$

13. Compute the convolutions $f * g$ of following functions f, g on $\mathbb{R}$ and then find the Fourier transform of $f * g$.

$$f(x) = \chi_a(x), g(x) = \chi_b(x)(x);$$

(Here $\chi_a = \chi_{[-a,a]}$ is the characteristic function of $[-a,a]$; generally $\chi_{[c,d]})(x)$, also denoted by $\mathbf{1}_{[c,d]}$, stands for the characteristic function of the interval $[c,d]$, namely it is 1 on the interval and 0 elsewhere).

$$f(x) = \chi_a(x), g = \frac{1}{x^2+1}$$

14. Compute $f * g$ and its Laplace transforms, where f, g are

$$f(t) = \chi_{[0,a]}(t), g(t) = \chi_{[0,b]}(t);$$

$$f(t) = H(t)e^{-t}, g(t) = H(t)\sin t.$$

Answers/Hints:

4. Apply Parsevals identity to Table1, Entry 12 and Entry 6 (or use the convergence theorem for the Entry 4).

5. Apply Plancherel theorem to the Fourier transform of χ_a.

6. (a) and (c) are not allowed, since the functions are not continuous, $f'(\theta)$ is not a well-defined function. (b) is a valid argument.

7. (a.) Point-wise convergence, and the sum is not continuous. (b). Same as (a). (c) C^1-function. (d) C^∞-function.

8. (a) $u(x,t) = e^{-(\frac{3\pi}{l})^2 t} \sin \frac{3\pi x}{l} - e^{-(\frac{\pi}{l})^2 t} \sin \frac{\pi x}{l}$, (b) $u(x,t) = \frac{1}{2}(l-x)x + e^{-(\frac{3\pi}{l})^2 t} \sin \frac{3\pi x}{l} - e^{-(\frac{\pi}{l})^2 t} \sin \frac{\pi x}{l} - \frac{4l^3}{\pi^3} \sum_1^\infty (2n-1)^{-3} e^{-(\frac{(2n-1)\pi}{l})^2 t} \sin \frac{(2n-1)\pi x}{l}$.

9. $u(x,t) = 1 - \frac{4}{\pi} \sum_1^\infty (2n-1)^{-1} \cos \frac{(2n-1)\pi t}{l} \sin \frac{(2n-1)\pi x}{l} + \frac{2l}{\pi} \sum_1^\infty (-1)^{n+1} n^{-1} \cos \frac{n\pi t}{l} \sin \frac{n\pi x}{l}$

10. (a) $u(x,y) = \frac{8l^3}{\pi^3} \sum_1^\infty (2n-1)^{-3} \sin \frac{(2n-1)\pi x}{l} \cosh \frac{(2n-1)\pi t}{l}$. (b) $u(r,\theta) = 2\sum_1^\infty r^n n^{-1} \sin n\theta$ (c) $u(r,\theta) = \frac{1}{4} r \sin\theta + \frac{1}{4} r^3 \sin 3\theta$

11.

12. Write $f = x^2 e^{-x}, u_1 = e^{-x}, u_2 = xe^{-x}$. The orthogonal projection $a_1 u_1 + a_2 u_2$ of f onto the subspace spaned by u_1, u_2 is the best approximation. Namely $f = f_0 + au_1 + bu_2$ where f_0 is orthogonal to the span. Taking the inner product with u_1, u_2 we get a linear system

$$\langle f, u_1 \rangle = a_1 \langle u_1, u_1 \rangle + a_2 \langle u_2, u_1 \rangle,$$

$$\langle f, u_2 \rangle = a_1 \langle u_1, u_2 \rangle + a_2 \langle u_2, u_2 \rangle.$$

$a = a_1, b = a_2$ can then be found by solving the matrix equation

$$(a_1, a_2) = (-1/2, 2).$$

13. Say (and can assume) $a < b$.

$$\chi_a * \chi_b(x) = \text{length of the intersection of the intervals } (-a,a) \text{ and } (x-b, x+b)$$
$$= \begin{cases} 2a, & |x| < b-a \\ a+b-|x|, & b-a < |x| < b+a \\ \\ 0, & |x| > a+b \end{cases}.$$

$$\chi_a * f(x) = \arctan(x+a) - \arctan(x-a). \quad f(x) = \frac{1}{x^2+1}.$$

(The Fourier transforms are obtained afterwards by using the general rule.)

14. (This is a bit harder. It can also be computated using the Laplace transform.) Say $a \leq b$.

$$\chi_{[0,a]} * \chi_{[0,b]}(t) = \text{length of the intersection of the three intervals } [0,t],\ [0,a] \text{ and } [t-b,t]$$
$$= \begin{cases} t, & t < a \\ a & a < t < b \\ & \\ a+b-t, & b < t < a+b \\ 0, & t > a+b \end{cases}$$

$$f * g(t) = -\frac{1}{2}e^{-t} + \sin t - \cos t.$$